NONLINEAR COHERENT STATES AND EHRENFEST TIME FOR SCHRÖDINGER EQUATION

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ABSTRACT. We consider the propagation of wave packets for the nonlinear Schrödinger equation, in the semi-classical limit. We establish the existence of a critical size for the initial data, in terms of the Planck constant: if the initial data are too small, the nonlinearity is negligible up to the Ehrenfest time. If the initial data have the critical size, then at leading order the wave function propagates like a coherent state whose envelope is given by a nonlinear equation, up to a time of the same order as the Ehrenfest time. We also prove a nonlinear superposition principle for these nonlinear wave packets.

1. Introduction

We consider semi-classical limit $\varepsilon \to 0$ for the nonlinear Schrödinger equation

$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \quad (t,x)\in\mathbf{R}_+\times\mathbf{R}^d, \\ \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0, \end{cases} \tag{1.1}$$

with $\lambda \in \mathbf{R}$, $d \geqslant 1$. The nonlinearity is energy subcritical ($\sigma < 2/(d-2)$ if $d \geqslant 3$). This equation arises for instance as a model for Bose–Einstein Condensation, where, among other possibilities, V may be exactly a harmonic potential, or a truncated harmonic potential (hence not exactly quadratic); see e.g. [13, 26].

Assuming that V is smooth and subquadratic (this notion is made precise below, see Assumption 1.1), we know that for each $\varepsilon > 0$, (1.1) has a unique global solution in the energy space

$$\Sigma = \left\{ f \in H^1(\mathbf{R}^d), \ \ x \mapsto |x| f(x) \in L^2(\mathbf{R}^d) \right\},$$

provided $\psi_0^\varepsilon \in \Sigma$ and, either $\sigma < 2/d$, or ($\sigma \geqslant 2/d$ and $\lambda \geqslant 0$), while if $\lambda \geqslant 2/d$ and $\lambda < 0$, finite time blow-up may occur; see [7]. We assume that the initial data ψ_0^ε is a localized wave packet of the form

$$\psi_0^\varepsilon(x) = \varepsilon^\beta \times \varepsilon^{-d/4} a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) e^{i(x-x_0)\cdot\xi_0/\varepsilon}, \quad a \in \mathcal{S}(\mathbf{R}^d), \quad x_0, \xi_0 \in \mathbf{R}^d. \tag{1.2}$$

Such data, which are called *semi-classical wave packets* (or *coherent states*), have been extensively studied in the linear case (see e.g. [4, 10, 11, 33, 35]). In particular, Gaussian wave packets are used in numerical simulation of quantum chemistry like Initial Value Representations methods. On this subject, the reader can refer to the recent papers [36, 37, 39] where overview and references on the topics can be found. These methods rely on the fact that if the data is a wave packet, then the solution of the linear equation ($\lambda = 0$) associated with (1.1) still is a wave packet at leading order up to times of order $C \log\left(\frac{1}{\varepsilon}\right)$: such a large (as $\varepsilon \to 0$) time is called

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Ehrenfest time, see e.g. [3, 22, 23]. Our aim here is to investigate what remains of these facts in the nonlinear case ($\lambda \neq 0$).

In the present nonlinear setting, a new parameter has to be considered: the size of the initial data, hence the factor ε^β in (1.2). The goal of this paper is to justify a notion of criticality for β: for $\beta > \beta_c := 1/(2\sigma) + d/4$, the initial data are too small to ignite the nonlinearity at leading order, and the leading order behavior of ψ^ε as $\varepsilon \to 0$ is the same as in the linear case $\lambda = 0$, up to Ehrenfest time. On the other hand, if $\beta = \beta_c$, the function ψ^ε is given at leading order by a wave packet whose envelope satisfies a *nonlinear* equation, up to a nonlinear analogue of the Ehrenfest time. We show moreover a nonlinear superposition principle: when the initial data is the sum of two wave packets of the form (1.2), then ψ^ε is approximated at leading order by the sum of the approximations obtained in the case of a single initial coherent state.

Up to changing ψ^ε to $\varepsilon^{-\beta}\psi^\varepsilon$, we may assume that the initial data are of order $\mathcal{O}(1)$ in $L^2(\mathbf{R}^d)$, and we consider

$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d, \\ \psi^\varepsilon(0,x) = \varepsilon^{-d/4} a\left(\dfrac{x-x_0}{\sqrt{\varepsilon}}\right) e^{i(x-x_0)\cdot\xi_0/\varepsilon}, \end{cases} \tag{1.3}$$

where $\alpha = 2\beta\sigma$.

1.1. **The linear case.** In this paragraph, we assume $\lambda = 0$:

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = \varepsilon^{-d/4} a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) e^{i(x-x_0)\cdot\xi_0/\varepsilon}. \tag{1.4}$$

The assumption we make on the external potential throughout this paper (even when $\lambda \neq 0$) is the following:

Assumption 1.1. *The external potential V is smooth, real-valued, and subquadratic:*

$$V \in C^\infty(\mathbf{R}^d;\mathbf{R}) \quad \text{and} \quad \partial_x^\gamma V \in L^\infty\left(\mathbf{R}^d\right), \quad \forall |\gamma| \geqslant 2.$$

Consider the classical trajectories associated with the Hamiltonian $\frac{|\xi|^2}{2} + V(x)$:

$$\dot{x}(t) = \xi(t), \;\; \dot{\xi}(t) = -\nabla V(x(t)); \quad x(0) = x_0, \; \xi(0) = \xi_0. \tag{1.5}$$

These trajectories satisfy

$$\frac{|\xi(t)|^2}{2} + V(x(t)) = \frac{|\xi_0|^2}{2} + V(x_0), \quad \forall t \in \mathbf{R}.$$

The fact that the potential is subquadratic implies that the trajectories grow at most exponentially in time.

Notation. For two positive numbers a^ε and b^ε, the notation $a^\varepsilon \lesssim b^\varepsilon$ means that there exists $C > 0$ *independent of* ε such that for all $\varepsilon \in]0,1]$, $a^\varepsilon \leqslant Cb^\varepsilon$.

Lemma 1.2. *Let $(x_0,\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d$. Under Assumption 1.1,* (1.5) *has a unique global, smooth solution $(x,\xi) \in C^\infty(\mathbf{R};\mathbf{R}^d)^2$. It grows at most exponentially:*

$$\exists C_0 > 0, \quad |x(t)| + |\xi(t)| \lesssim \mathrm{e}^{C_0 t}, \quad \forall t \in \mathbf{R}. \tag{1.6}$$

Sketch of the proof. We explain the exponential control only. We infer from (1.5) that x solves an Hamiltonian ordinary differential equation,

$$\ddot{x}(t) + \nabla V(x(t)) = 0.$$

Multiply this equation by $\dot{x}(t)$,

$$\frac{d}{dt}\left((\dot{x})^2 + V(x(t))\right) = 0,$$

and notice that in view of Assumption 1.1, $V(y) \lesssim \langle y \rangle^2$:

$$\dot{x}(t) \lesssim \langle x(t) \rangle,$$

and the estimate follows. □

Remark 1.3. The case $V(x) = -|x|^2$ shows that the result of Lemma 1.2 is sharp.

We associate with these trajectories the *classical action*

$$S(t) = \int_0^t \left(\frac{1}{2}|\xi(s)|^2 - V(x(s))\right) ds. \tag{1.7}$$

We observe that if we change the unknown function ψ^ε to u^ε by

$$\psi^\varepsilon(t,x) = \varepsilon^{-d/4} u^\varepsilon\left(t, \frac{x-x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon}, \tag{1.8}$$

then, in terms of $u^\varepsilon = u^\varepsilon(t,y)$, (1.4) is equivalent

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = V^\varepsilon(t,y)u^\varepsilon \quad ; \quad u^\varepsilon(0,y) = a(y), \tag{1.9}$$

where the external time-dependent potential V^ε is given by

$$V^\varepsilon(t,y) = \frac{1}{\varepsilon}\left(V(x(t)+\sqrt{\varepsilon}y) - V(x(t)) - \sqrt{\varepsilon}\langle \nabla V(x(t)), y\rangle\right). \tag{1.10}$$

This expression reveals the first terms of the Taylor expansion of V about the point $x(t)$. Passing formally to the limit, V^ε converges to the Hessian of V at $x(t)$ evaluated at (y,y). One does not even need to pass to the limit if V is a polynomial of degree at most two: in that case, we see that the solution ψ^ε remains exactly a coherent state for all time. Let us denote by $Q(t)$ the symmetric matrix

$$Q(t) = \operatorname{Hess} V(x(t)).$$

It is well-known that if v solves

$$i\partial_t v + \frac{1}{2}\Delta v = \frac{1}{2}\langle Q(t)y, y\rangle v \quad ; \quad v(0,y) = a(y), \tag{1.11}$$

then the function

$$\varphi^\varepsilon_{\rm lin}(t,x) = \varepsilon^{-d/4} v\left(t, \frac{x-x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon} \tag{1.12}$$

approximates ψ^ε for large time in the sense that there exists $C>0$ independent of ε such that

$$\|\psi^\varepsilon(t,\cdot) - \varphi^\varepsilon_{\rm lin}(t,\cdot)\|_{L^2(\mathbf{R}^d)} \leqslant C\sqrt{\varepsilon}e^{Ct}.$$

See e.g. [3, 10, 11, 12, 21, 22, 23] and references therein. We give a short proof of this estimate, which can be considered as the initial step toward the nonlinear

analysis which is presented in the next paragraph. We first notice that since V is subquadratic, we have the following pointwise estimate:

$$\left| V^\varepsilon(t,y) - \frac{1}{2}\langle Q(t)y,y\rangle \right| \leqslant \underline{C}\sqrt{\varepsilon}|y|^3, \tag{1.13}$$

for some constant $\underline{C}$ independent of t. The error $r^\varepsilon_{\text{lin}} = u^\varepsilon - v$ satisfies

$$i\partial_t r^\varepsilon_{\text{lin}} + \frac{1}{2}\Delta r^\varepsilon_{\text{lin}} = V^\varepsilon u^\varepsilon - \frac{1}{2}\langle Q(t)y,y\rangle v = V^\varepsilon r^\varepsilon_{\text{lin}} + \left(V^\varepsilon - \frac{1}{2}\langle Q(t)y,y\rangle \right) v,$$

along with the initial value $r^\varepsilon_{\text{lin}|t=0} = 0$. Since V^ε is real-valued, the classical energy estimate for Schrödinger equations yields, in view of (1.13),

$$\|r^\varepsilon_{\text{lin}}\|_{L^\infty([0,t];L^2(\mathbf{R}^d))} \lesssim \sqrt{\varepsilon}\int_0^t \||y|^3 v(\tau,y)\|_{L^2(\mathbf{R}^d)}d\tau.$$

Since Q is bounded (V is subquadratic), we have the control

$$\||y|^3 v(\tau,y)\|_{L^2(\mathbf{R}^d)} \leqslant Ce^{C\tau}$$

for some constant $C>0$; see Proposition 2.1 below. We then have to notice that the wave packet scaling is L^2-unitary:

$$\|\psi^\varepsilon(t,\cdot) - \varphi^\varepsilon_{\text{lin}}(t,\cdot)\|_{L^2(\mathbf{R}^d)} = \|u^\varepsilon(t,\cdot) - v(t,\cdot)\|_{L^2(\mathbf{R}^d)}.$$

To summarize, we have:

Lemma 1.4. *Let $d \geqslant 1$ and $a \in \mathcal{S}(\mathbf{R}^d)$. There exists $C>0$ independent of ε such that*

$$\|\psi^\varepsilon(t,\cdot) - \varphi^\varepsilon_{\text{lin}}(t,\cdot)\|_{L^2(\mathbf{R}^d)} \leqslant C\sqrt{\varepsilon}e^{Ct}. \tag{1.14}$$

In particular, there exists $c>0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t,\cdot) - \varphi^\varepsilon_{\text{lin}}(t,\cdot)\|_{L^2(\mathbf{R}^d)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

1.2. **The nonlinear case.** We now consider the nonlinear situation $\lambda \neq 0$. Resuming the same change of unknown function (1.8), then adapting the above computation leads to

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = V^\varepsilon u^\varepsilon + \lambda\varepsilon^{\alpha-\alpha_c}|u^\varepsilon|^{2\sigma}u^\varepsilon, \tag{1.15}$$

where V^ε is given by (1.10) as in the linear case, and

$$\alpha_c = 1 + \frac{d\sigma}{2}. \tag{1.16}$$

The real number α_c appears as a critical exponent. In the case $\alpha > \alpha_c$, we can approximate the nonlinear solution u^ε by the same function v as in the linear case, given by (1.11). The space Σ will turn out to be quite natural for energy estimates. Introduce the operators

$$A^\varepsilon(t) = \sqrt{\varepsilon}\nabla - i\frac{\xi(t)}{\sqrt{\varepsilon}} \quad ; \quad B^\varepsilon(t) = \frac{x - x(t)}{\sqrt{\varepsilon}}.$$

Note that A and B are essentially ∇ and x, up to the wave packet scaling, in the moving frame. From this point of view, our energy space is quite different from the one associated with the Lyapounov functional considered in [14], and more related

to the one considered in [25], since we pay attention to the localization of the wave packet, through B^ε. For $f \in \Sigma$, we set

$$\|f\|_{\mathcal{H}} = \|f\|_{L^2(\mathbf{R}^d)} + \|A^\varepsilon(t)f\|_{L^2(\mathbf{R}^d)} + \|B^\varepsilon(t)f\|_{L^2(\mathbf{R}^d)},$$

a notation where we do not emphasize the fact that this norm depends on ε and t.

Proposition 1.5. *Let $d \geqslant 1$, $a \in \mathcal{S}(\mathbf{R}^d)$. Suppose that $\alpha > \alpha_c$. There exist $C, C_1 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$,*

$$\|\psi^\varepsilon(t) - \varphi^\varepsilon_{\rm lin}(t)\|_{\mathcal{H}} \lesssim \varepsilon^\gamma e^{C_1 t}, \quad 0 \leqslant t \leqslant C \log \frac{1}{\varepsilon}, \quad \text{where } \gamma = \min\left(\frac{1}{2}, \alpha - \alpha_c\right).$$

In particular, there exists $c > 0$ independent of ε such that

$$\sup_{0 \leqslant t \leqslant c \log \frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi^\varepsilon_{\rm lin}(t)\|_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

The proof is more complicated than in the linear case (see §2). The solution of (1.3) is linearizable in the sense of [18] (see also [8]), up to an Ehrenfest time.

In the critical case $\alpha = \alpha_c$ with $\lambda \neq 0$, the solution of (1.3) is no longer linearizable. Indeed, passing formally to the limit $\varepsilon \to 0$, Equation (1.15) becomes

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\langle Q(t)y, y\rangle u + \lambda |u|^{2\sigma} u \quad ; \quad u(0,y) = a(y). \tag{1.17}$$

Remark 1.6 (Complete integrability). The cubic one-dimensional case $d = \sigma = 1$ is special: if $\dot{Q} = 0$, then (1.17) is completely integrable ([1]). However, if $\dot{Q} \neq 0$, there exists no Lax pair when the nonlinearity is autonomous as in (1.17); see [30, 38]. Note also that if $\dot{Q} = 0$, then $u^\varepsilon = u$ for all time.

As in the linear case, we note that if V is exactly a polynomial of degree at most two, then u is actually equal to u^ε for all ε. The global well-posedness for (1.17) has been established in [7]. We first prove that u yields a good approximation for u^ε on bounded time intervals:

Proposition 1.7. *Let $d \geqslant 1$, $\sigma > 0$ with $\sigma < 2/(d-2)$ if $d \geqslant 3$, and $a \in \mathcal{S}(\mathbf{R}^d)$. Let $u \in C(\mathbf{R}; \Sigma)$ be the solution to (1.17), and let*

$$\varphi^\varepsilon(t,x) = \varepsilon^{-d/4} u\left(t, \frac{x - x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t) + \xi(t)\cdot(x - x(t)))/\varepsilon}. \tag{1.18}$$

For all $T > 0$ (independent of $\varepsilon > 0$), we have

$$\sup_{0 \leqslant t \leqslant T} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} = \mathcal{O}\left(\sqrt{\varepsilon}\right).$$

If in addition $\sigma > 1/2$,

$$\sup_{0 \leqslant t \leqslant T} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{\mathcal{H}} = \mathcal{O}\left(\sqrt{\varepsilon}\right).$$

Remark 1.8. The presence of u, which solves a nonlinear equation, clearly shows that the nonlinearity modifies the coherent state at leading order. Note however that the Wigner measure of ψ^ε (see e.g. [19, 32]) is not affected by the nonlinearity:

$$\begin{aligned} w(t,x,\xi) &= \|u(t)\|^2_{L^2(\mathbf{R}^d)} \delta\left(x - x(t)\right) \otimes \delta\left(\xi - \xi(t)\right) \\ &= \|a\|^2_{L^2(\mathbf{R}^d)} \delta\left(x - x(t)\right) \otimes \delta\left(\xi - \xi(t)\right). \end{aligned}$$

The Wigner measure remains the same because the nonlinearity alters only the envelope of the coherent state, not its center in phase space.

Remark 1.9 (Supercritical case). Consider the case $\alpha < \alpha_c$, and assume for instance $V = 0$. Resuming the scaling (1.8), Equation (1.15) becomes

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = \lambda \varepsilon^{\alpha-\alpha_c}|u^\varepsilon|^{2\sigma}u^\varepsilon.$$

At time $t = 0$, u^ε is independent of ε: $u^\varepsilon_{|t=0} = a$. Setting $\hbar^2 = \varepsilon^{\alpha_c-\alpha}$ and changing the time variable to $s = t/\hbar$, the problem reads

$$i\hbar\partial_s u^\hbar + \frac{\hbar^2}{2}\Delta u^\hbar = |u^\hbar|^{2\sigma}u^\hbar \quad ; \quad u^\hbar(0,x) = a(x). \tag{1.19}$$

Therefore, to understand the asymptotic behavior of u as $\varepsilon \to 0$ (or equivalently, as $\hbar \to 0$) for $t \in [0,T]$, we need to understand the large time ($s \in [0, T/\hbar]$) behavior in (1.19). This corresponds to a large time semi-classical limit in the (supercritical) WKB regime. Describing this behavior is extremely delicate, and still an open problem; see [6].

In order to prove the validity of the approximation on large time intervals, we introduce the following notion:

Definition 1.10. *Let $u \in C(\mathbf{R};\Sigma)$ be a solution to* (1.17)*, and $k \in \mathbf{N}$. We say that $(Exp)_k$ is satisfied if there exists $C = C(k)$ such that*

$$\forall \alpha,\beta \in \mathbf{N}^d,\ |\alpha| + |\beta| \leqslant k, \quad \left\| x^\alpha \partial_x^\beta u(t)\right\|_{L^2(\mathbf{R}^d)} \lesssim e^{Ct}.$$

Note that reasonably, to establish $(Exp)_k$, the larger the k, the smoother the nonlinearity $z \mapsto |z|^{2\sigma}z$ has to be. For simplicity, we shall now assume $\sigma \in \mathbf{N}$.

Proposition 1.11 (From [7])**.** *Let $d \leqslant 3$, $\sigma \in \mathbf{N}$ with $\sigma = 1$ if $d = 3$, $a \in \mathcal{S}(\mathbf{R}^d)$ and $k \in \mathbf{N}$. Then $(Exp)_k$ is satisfied (at least) in the following cases:*

- *$\sigma = d = 1$ and $\lambda \in \mathbf{R}$ (cubic one-dimensional case).*
- *$\sigma \geqslant 2/d$, $\lambda > 0$ and $Q(t)$ is diagonal with eigenvalues $\omega_j(t) \leqslant 0$.*
- *$\sigma \geqslant 2/d$, $\lambda > 0$ and $Q(t)$ is compactly supported.*

It is very likely that this result remains valid under more general assumptions (see in particular [7, §6.2] for the case $\sigma = 2/d$). Yet, we have not been able to prove it. Let us comment a bit on these three cases. The first case is the most general one concerning the potential V and the classical trajectory $x(t)$: the only assumption carries over the nonlinearity (the important aspect is that it is L^2-subcritical). The other two cases concern L^2-critical or supercritical defocusing nonlinearities. In the second case, V is required to be concave (along the classical trajectory), and the last case corresponds for instance to a compactly supported HessV, when the classical trajectory is not trapped. In this last case, we have actually better than an exponential decay: Sobolev norms are bounded, and momenta grow algebraically in time. The following result could be improved in this case.

Theorem 1.12. *Let $a \in \mathcal{S}(\mathbf{R}^d)$. If $(Exp)_4$ is satisfied, then there exist $C, C_2 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0,\varepsilon_0]$,*

$$\|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{\mathcal{H}} \lesssim \sqrt{\varepsilon}\exp\left(\exp(C_2 t)\right), \quad 0 \leqslant t \leqslant C\log\log\frac{1}{\varepsilon}.$$

In particular, there exists $c > 0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{\mathcal{H}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

In the one-dimensional cubic case, this result can be improved on two aspects. First, we can prove a long time asymptotics in L^2 provided $(Exp)_3$ is satisfied. More important is the fact that we obtain an asymptotics up to an Ehrenfest time:

Theorem 1.13. *Assume $d = \sigma = 1$, and let $a \in \mathcal{S}(\mathbf{R})$. If $(Exp)_3$ is satisfied, then there exist $C, C_3 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$,*

$$\|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbf{R})} \lesssim \sqrt{\varepsilon}\exp(C_3 t), \quad 0 \leqslant t \leqslant C \log\frac{1}{\varepsilon}.$$

In particular, there exists $c > 0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbf{R})} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

If in addition $(Exp)_4$ is satisfied, then for the same constants as above,

$$\|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{\mathcal{H}} \lesssim \sqrt{\varepsilon}\exp(C_3 t), \quad 0 \leqslant t \leqslant C \log\frac{1}{\varepsilon},$$

and

$$\sup_{0\leqslant t\leqslant c\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{\mathcal{H}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The technical reason which explains the differences between Theorem 1.12 and Theorem 1.13 is that the one-dimensional cubic case is L^2-subcritical. This aspect has several consequences regarding the Strichartz estimates we use in the course of the proof.

These nonlinear results are to be compared with previous ones concerning the interaction between a linear dynamics (classical trajectories) and nonlinear effects. Consider the WKB regime

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = \varepsilon^{\widetilde{\beta}}a(x)e^{ix\cdot\xi_0/\varepsilon}, \tag{1.20}$$

with V satisfying Assumption 1.1. Like above, it is equivalent, up to a rescaling, to

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon^{\widetilde{\alpha}}|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon(0,x) = a(x)e^{ix\cdot\xi_0/\varepsilon},$$

with $\widetilde{\alpha} = 2\sigma\widetilde{\beta}$. The critical value in this regime is $\widetilde{\alpha}_c = 1$ (see [6]). In (1.20), this corresponds to initial data of order $\varepsilon^{1/(2\sigma)}$ in L^∞, like in the present case of wave packets. However, the critical nonlinear effects are very different in the case of (1.20). The following asymptotics holds in $L^2(\mathbf{R}^d)$ (see [6]):

$$\psi^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} a(t,x)e^{ig(t,x)}e^{i\phi(t,x)/\varepsilon},$$

as long as the phase ϕ, solution to the Hamilton–Jacobi equation

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + V = 0 \quad ; \quad \phi(0,x) = x\cdot\xi_0,$$

remains smooth. More general initial phases are actually allowed: we consider an initial phase linear in x for the comparison with (1.3). The amplitude a solves a *linear* transport equation: at leading order, nonlinear effects show up through the phase modulation g (which depends on λ and σ). This result calls for at least two comments. First, this nonlinear effect is rather weak: for instance, it does not affect the main quadratic observables at leading order, $|\psi^\varepsilon|^2$ (position density) and $\varepsilon\,\mathrm{Im}\,\overline{\psi}^\varepsilon\nabla\psi^\varepsilon$ (current density). In the case of (1.3), the profile equation is, in a sense, more nonlinear, even though in both cases, Wigner measures are not affected by

the critical nonlinearity. Second, the validity of WKB analysis is limited in general, even if V is a polynomial. If $V=0$, $\phi(t,x)=x\cdot\xi_0-t|\xi_0|^2/2$ is smooth for all time, $a(t,x)=a_0(x-t\xi_0)$ remains bounded, and the asymptotics can be justified up to Ehrenfest time, by simply resuming the proof given in [6]. If $V(x)=E\cdot x$, Avron–Herbst formula shows that this case is essentially the same as $V=0$. On the other hand, if $V(x)=\omega^2|x|^2/2$, classical trajectories in (1.5) are explicit:

$$x(t)=x_0\cos(\omega t)+\xi_0\frac{\sin(\omega t)}{\omega}.$$

They all meet at ξ_0/ω at time $t_*=\pi/(2\omega)$: the phase ϕ becomes singular as $t\to t_*$, and WKB analysis ceases to be valid, while the wave packets approach yields an exact result for all time in such a case.

In [5, 14, 17, 24, 25, 28, 29], the authors have considered a similar problem, in a different regime though:

$$i\varepsilon\partial_t\psi^\varepsilon+\frac{\varepsilon^2}{2}\Delta\psi^\varepsilon=V(x)\psi^\varepsilon-|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon\quad;\quad\psi^\varepsilon(0,x)=Q\left(\frac{x-x_0}{\varepsilon}\right)e^{i\xi_0\cdot x/\varepsilon},\tag{1.21}$$

where Q is a ground state, solution to a nonlinear elliptic equation. They prove, with some precision depending on the papers:

$$\psi^\varepsilon(t,x)\underset{\varepsilon\to0}{\sim}Q\left(\frac{x-x(t)}{\varepsilon}\right)e^{i\xi(t)\cdot x/\varepsilon+i\theta^\varepsilon(t)},\quad\theta^\varepsilon(t)\in\mathbf{R}.$$

As pointed out in [24], such results may be extended to an Ehrenfest time. An important difference with our paper must be emphasized, besides the scaling: the particular initial data makes it possible to rely on rigidity properties of the solitary waves, which do not hold for general profiles. In [9], some results concerning a defocusing equation with more general initial profiles are proved (or cited), in the same scaling as in (1.21): however, it seems that unless V is a polynomial of degree at most two, only partial results are available then (that is, on relatively small time intervals). Finally, even when $\partial^\gamma V=$ for all $|\gamma|\geqslant 3$, the time intervals on which some asymptotic results are proved must be independent of ε.

1.3. **Nonlinear superposition.** We still suppose $\alpha=\alpha_c$. For simplicity, in this paragraph, we assume that σ is an integer: this is compatible with the fact that the nonlinearity is energy-subcritical only if $d\leqslant 3$. We consider initial data corresponding to the superposition of two wave packets:

$$\psi^\varepsilon(0,x)=\varepsilon^{-d/4}a_1\left(\frac{x-x_1}{\sqrt{\varepsilon}}\right)e^{i(x-x_1)\cdot\xi_1/\varepsilon}+\varepsilon^{-d/4}a_2\left(\frac{x-x_2}{\sqrt{\varepsilon}}\right)e^{i(x-x_2)\cdot\xi_2/\varepsilon},$$

with $a_1,a_2\in\mathcal{S}(\mathbf{R})$, $(x_1,\xi_1),(x_2,\xi_2)\in\mathbf{R}^2$, and $(x_1,\xi_1)\neq(x_2,\xi_2)$. For $j\in\{1,2\}$, $(x_j(t),\xi_j(t))$ are the classical trajectories solutions to (1.5) with initial data (x_j,ξ_j). We denote by S_j the action associated with $(x_j(t),\xi_j(t))$ by (1.7) and by u_j the solution of (1.17) for the curve $x_j(t)$ and with initial data a_j. We consider φ_j^ε associated by (1.18) with u_j,x_j,ξ_j,S_j, and $\psi^\varepsilon\in C(\mathbf{R};\Sigma)$ solution to (1.3) with $\alpha=\alpha_c$ and the above initial data.

The functional setting used to describe the function ψ^ε must be changed in the case of two initial wave packets: recall that $\mathcal{H}$ is defined through A^ε and B^ε, which are related to the Hamiltonian flow. The geometric meaning of A^ε and B^ε becomes irrelevant in the case of two wave packets. Instead, we use norms on Σ whose geometric meaning is weaker, since essentially, they reflect the fact that we

consider ε-oscillatory functions, which remain somehow localized in space (before Ehrenfest time):

$$\|f\|_{\Sigma_\varepsilon} = \|f\|_{L^2(\mathbf{R}^d)} + \|\varepsilon\nabla f\|_{L^2(\mathbf{R}^d)} + \|xf\|_{L^2(\mathbf{R}^d)}.$$

For finite time, we have:

Proposition 1.14. *Let $d \leqslant 3$, $\sigma \in \mathbf{N}$ ($\sigma = 1$ if $d = 3$), and $a_1, a_2 \in \mathcal{S}(\mathbf{R}^d)$. For all $T > 0$ (independent of ε), we have, for all $\gamma < 1/2$:*

$$\sup_{0\leqslant t\leqslant T} \|\psi^\varepsilon(t) - \varphi_1^\varepsilon(t) - \varphi_2^\varepsilon(t)\|_{\Sigma_\varepsilon} = \mathcal{O}\left(\varepsilon^\gamma\right).$$

Besides, nonlinear superposition holds for large time (at least) in the one-dimensional case, if the points (x_1, ξ_1) and (x_2, ξ_2) have different energies.

Theorem 1.15. *Assume that $d = 1$, σ is an integer, and let $a_1, a_2 \in \mathcal{S}(\mathbf{R})$. Suppose that $E_1 \neq E_2$, where*

$$E_j = \frac{\xi_j^2}{2} + V\left(x_j\right).$$

Suppose that $(Exp)_k$ is satisfied for some $k \geqslant 4$ (for u_1 and u_2).
1. *There exist $C, C_3 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$,*

$$\|\psi^\varepsilon(t) - \varphi_1(t)^\varepsilon - \varphi_2^\varepsilon(t)\|_{\Sigma_\varepsilon} \lesssim \varepsilon^\gamma \exp\left(\exp(C_3 t)\right), \quad 0 \leqslant t \leqslant C \log\log\frac{1}{\varepsilon},$$

with $\gamma = \frac{k-2}{2k-2}$. In particular, there exists $c > 0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi_1(t)^\varepsilon - \varphi_2^\varepsilon(t)\|_{\Sigma_\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

2. *Suppose in addition that $\sigma = 1$. There exist $C, C_4 > 0$ independent of ε, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$,*

$$\|\psi^\varepsilon(t) - \varphi_1(t)^\varepsilon - \varphi_2^\varepsilon(t)\|_{\Sigma_\varepsilon} \lesssim \varepsilon^\gamma e^{C_4 t}, \quad 0 \leqslant t \leqslant C\log\frac{1}{\varepsilon}, \quad \text{with } \gamma = \frac{k-2}{2k-2}.$$

In particular, there exists $c > 0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi_1(t)^\varepsilon - \varphi_2^\varepsilon(t)\|_{\Sigma_\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

It is interesting to see that even though the profiles are nonlinear, the superposition principle, which is a property of linear equations, still holds. There are many other such nonlinear superposition principles in the literature, and we cannot mention them all.

This result is to be compared with those in [31] (see also references therein), for several reasons. In [31], the authors construct a solution for the three-dimensional Schrödinger–Poisson system which behaves, in $H^1(\mathbf{R}^3)$ and asymptotically for large time, like the sum of two ground state solitary waves. The two solitary waves are centered, in the phase space, at the solution of a two-body problem: unlike what happens in our case, there exists an interaction between the trajectories, due to the fact that the Poisson potential is long range. In our case, the long range aspect of the nonlinearity (when $d = \sigma = 1$; see [34]) does not have such a consequence: we will see that the key point in the proof of the above two results is the fact that in the wave packet scaling, the two functions φ_1^ε and φ_2^ε do not interact at leading order in the limit $\varepsilon \to 0$: the nonlinear effects concentrate on the profiles, along the classical trajectories, and it turns out that these trajectories do not meet "too

much". In [8], another nonlinear superposition principle was proved, in the scaling of (1.21). However, in [8], nonlinear effects were localized in space *and* time, so most of the time, the nonlinear superposition was actually a linear one.

1.4. **Outline of the paper.** In Section 2, we first analyze the linearizable case and prove Proposition 1.5 after a short analysis of the linear case. Then, in Section 3, we recall basic facts about Strichartz estimates in this semi-classical framework and prove the consistency of our approximation on bounded time intervals. Theorem 1.12 is proved in Section 4. Finally, Section 5 is focused on the one-dimensional cubic case and Section 6 on the analysis of the nonlinear superposition.

Notation. Throughout the paper, in the expression e^{Ct}, the constant C will denote a constant independent of t which may change from one line to the other.

2. The linearizable case

In this section, we assume $\alpha > \alpha_c$ and we prove Proposition 1.5. We first recall estimates in the linear case $\lambda = 0$ which are more precise than in §1.1.

2.1. **The linear case.** We suppose here $\lambda = 0$. The first remark concerns the properties of the profile v. It is not difficult to prove the following proposition.

Proposition 2.1. *Let $d \geqslant 1$ and $a \in \mathcal{S}(\mathbf{R}^d)$. For all $k \in \mathbf{N}$, there exists $C > 0$ such that the solution v to* (1.11) *satisfies*

$$\forall \alpha, \beta \in \mathbf{N}^d, \ \ |\alpha| + |\beta| \leqslant k, \ \ \left\| x^\alpha \partial_x^\beta v(t) \right\|_{L^2(\mathbf{R}^d)} \lesssim e^{Ct}.$$

A general proof of Proposition 2.1 is given for instance in [7, §6.1]. Let us now consider $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon_{\rm lin}$. We have $w^\varepsilon(0) = 0$ and

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = V(x) w^\varepsilon - \left(V(x) - T_2(x, x(t))\right)\varphi^\varepsilon_{\rm lin},$$

where T_2 corresponds to a second order Taylor approximation:

$$T_2(x,a) := V(a) + \langle \nabla V(a), x-a\rangle + \frac{1}{2}\left\langle \mathrm{Hess}V(a)(x-a), x-a\right\rangle.$$

We have seen in §1.1 that the standard L^2 estimate for Schrödinger equations yields

$$\|w^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} \lesssim \sqrt{\varepsilon}\|y^3 v(t)\|_{L^2(\mathbf{R}^d)} \lesssim \sqrt{\varepsilon}e^{Ct}.$$

In order to analyze the convergence in Σ, we can write

$$\left(i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta - V(x)\right)(\varepsilon\nabla w^\varepsilon) = \varepsilon \nabla V w^\varepsilon - \varepsilon\nabla L^\varepsilon,$$

$$\left(i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta - V(x)\right)(x w^\varepsilon) = \left[\frac{\varepsilon^2}{2}\Delta, x\right] w^\varepsilon - xL^\varepsilon = \varepsilon^2\nabla w^\varepsilon - xL^\varepsilon,$$

where

$$L^\varepsilon(t,x) := \left(V(x) - T_2(x,x(t))\right)\varphi^\varepsilon_{\rm lin}(t,x). \tag{2.1}$$

Typically if $d=1$,

$$\begin{aligned} L^\varepsilon(t,x) &= \frac{1}{2}\left(x - x(t)\right)^3 \varphi^\varepsilon_{\rm lin}(t,x)\int_0^1 V'''\left(x(t) + \theta\left(x - x(t)\right)\right)\theta^2 d\theta \\ &= \frac{\left(x-x(t)\right)^3}{2\varepsilon^{1/4}} e^{-i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon} v\left(t, \frac{x-x(t)}{\sqrt{\varepsilon}}\right) I\left(x, x(t)\right), \end{aligned}$$

where

$$I\left(x,x(t)\right)=\int_0^1 V^{'''}\left(x(t)+\theta\left(x-x(t)\right)\right)\theta^2 d\theta.$$

Energy estimates make it possible to show

$$\|\varepsilon\nabla w^\varepsilon(t)\|_{L^2(\mathbf{R}^d)}+\|xw^\varepsilon(t)\|_{L^2(\mathbf{R}^d)}\lesssim\sqrt{\varepsilon}e^{Ct}.$$

However, the operators A^ε and B^ε defined in the introduction yield more precise results. For instance, $\|\varepsilon\nabla\varphi_{\rm lin}\|_{L^2}$ is of order $\mathcal{O}(1)$ exactly, because of the phase factor in (1.12). We note the formula

$$A^\varepsilon(t)=\sqrt{\varepsilon}\nabla-i\frac{\xi(t)}{\sqrt{\varepsilon}}=\sqrt{\varepsilon}e^{i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon}\nabla\left(e^{-i(S(t)+\xi(t)\cdot(x-x(t)))/\varepsilon}\cdot\right),\tag{2.2}$$

so for instance $\|A^\varepsilon(t)\varphi_{\rm lin}\|_{L^2}$ is of order $\mathcal{O}(1)$: morally, we have gained a factor $\sqrt{\varepsilon}$.

Lemma 2.2. *The operators A^ε and B^ε, defined by*

$$A^\varepsilon(t)=\sqrt{\varepsilon}\nabla-i\frac{\xi(t)}{\sqrt{\varepsilon}}\quad;\quad B^\varepsilon(t)=\frac{x-x(t)}{\sqrt{\varepsilon}},$$

satisfy the commutation relations:

$$\left[i\varepsilon\partial_t+\frac{\varepsilon^2}{2}\Delta-V,A^\varepsilon(t)\right]=\sqrt{\varepsilon}\left(\nabla V(x)-\nabla V\left(x(t)\right)\right),$$
$$\left[i\varepsilon\partial_t+\frac{\varepsilon^2}{2}\Delta-V,B^\varepsilon(t)\right]=\varepsilon A^\varepsilon(t).$$

We can then write

$$\left(i\varepsilon\partial_t+\frac{\varepsilon^2}{2}\Delta-V(x)\right)A^\varepsilon(t)w^\varepsilon=\sqrt{\varepsilon}\left(\nabla V(x)-\nabla V\left(x(t)\right)\right)w^\varepsilon-A^\varepsilon(t)L^\varepsilon,$$
$$\left(i\varepsilon\partial_t+\frac{\varepsilon^2}{2}\Delta-V(x)\right)B^\varepsilon(t)w^\varepsilon=\varepsilon A^\varepsilon(t)w^\varepsilon-B^\varepsilon(t)L^\varepsilon.$$

In view of (2.2), we observe

$$\begin{aligned}\|A^\varepsilon(t)L^\varepsilon\|_{L^2(\mathbf{R}^d)}&\lesssim\varepsilon^{3/2}\|x^2v(t)\|_{L^2(\mathbf{R}^d)}+\varepsilon^{3/2}\|x^3\nabla v(t)\|_{L^2(\mathbf{R}^d)}+\varepsilon^2\|x^3v(t)\|_{L^2(\mathbf{R}^d)}\\&\lesssim\varepsilon^{3/2}e^{Ct},\end{aligned}$$

thanks to Lemma 2.1. Similarly,

$$\|B^\varepsilon(t)L^\varepsilon\|_{L^2(\mathbf{R}^d)}\lesssim\varepsilon^{3/2}e^{Ct}.$$

Since we have the pointwise estimate

$$\left|\sqrt{\varepsilon}\left(\nabla V(x)-\nabla V\left(x(t)\right)\right)w^\varepsilon\right|\lesssim\varepsilon\left|B^\varepsilon(t)w^\varepsilon\right|,$$

energy estimates yield

$$\|w^\varepsilon(t)\|_{\mathcal{H}}\lesssim\int_0^t\left(\|w^\varepsilon(s)\|_{\mathcal{H}}+\sqrt{\varepsilon}e^{Cs}\right)ds.$$

We conclude by Gronwall Lemma:

$$\|w^\varepsilon(t)\|_{\mathcal{H}}\lesssim\sqrt{\varepsilon}e^{Ct}.$$

We will see in the following subsection that the arguments are somehow more complicated in the nonlinear setting.

2.2. **Proof of Proposition 1.5.** We now assume $\lambda \neq 0$, and $\alpha > \alpha_c$. For the simplicity of the presentation, we give the detailed proof in the case $d = 1$ only.

We set again $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon_{\rm lin}$ and we write the equation satisfied by w^ε:

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2 w^\varepsilon = V(x)w^\varepsilon - \left(V(x) - T_2(x,x(t))\right)\varphi^\varepsilon_{\rm lin} + N^\varepsilon \quad ; \quad w^\varepsilon_{|t=0} = 0,$$

where the nonlinear source term is given by

$$N^\varepsilon = \lambda\varepsilon^\alpha|\varphi^\varepsilon_{\rm lin} + w^\varepsilon|^{2\sigma}(\varphi^\varepsilon_{\rm lin} + w^\varepsilon).$$

First, since $\lambda\varepsilon^\alpha|\varphi^\varepsilon_{\rm lin} + w^\varepsilon|^{2\sigma} \in \mathbf{R}$, the L^2 energy estimate for w^ε yields

$$\|w^\varepsilon(t)\|_{L^2(\mathbf{R})} \lesssim \frac{1}{\varepsilon}\|L^\varepsilon\|_{L^1([0,t];L^2(\mathbf{R}))} + \frac{1}{\varepsilon}\left\|\varepsilon^\alpha|\varphi^\varepsilon_{\rm lin} + w^\varepsilon|^{2\sigma}\varphi^\varepsilon_{\rm lin}\right\|_{L^1([0,t];L^2(\mathbf{R}))},$$

where we have kept the notation (2.1). The contribution of N^ε cannot be studied directly, since we do not know yet how to estimate w^ε: since w^ε will turn out to be small, we use a bootstrap argument.

Since we have $\|\varphi^\varepsilon_{\rm lin}(t)\|_{L^\infty(\mathbf{R})} = \varepsilon^{-1/4}\|v(t)\|_{L^\infty(\mathbf{R})}$, Proposition 2.1 and Sobolev embedding show that there exists $C_0 > 0$ such that

$$\|\varphi^\varepsilon_{\rm lin}(t)\|_{L^\infty(\mathbf{R})} \leqslant C_0\varepsilon^{-1/4}e^{C_0 t}, \quad \forall t \geqslant 0.$$

The bootstrap argument goes as follows. We suppose that for $t \in [0,\tau]$ we have

$$\|w^\varepsilon(t)\|_{L^\infty} \leqslant \varepsilon^{-1/4}e^{C_0 t}, \tag{2.3}$$

with the same constant C_0. Since $w^\varepsilon_{|t=0} = 0$ and $\psi^\varepsilon \in C(\mathbf{R};\Sigma)$, there exists $\tau^\varepsilon > 0$ (*a priori* depending on ε) such that (2.3) holds on $[0,\tau^\varepsilon]$. So long as (2.3) holds,

$$\left\|\varepsilon^\alpha|\varphi^\varepsilon_{\rm lin} + w^\varepsilon|^{2\sigma}\varphi^\varepsilon_{\rm lin}\right\|_{L^2(\mathbf{R})} \lesssim \varepsilon^{\alpha-\sigma/2}\|a\|_{L^2(\mathbf{R})}e^{2\sigma C_0 t}.$$

We infer

$$\|w^\varepsilon(t)\|_{L^2(\mathbf{R})} \lesssim \sqrt{\varepsilon}e^{Ct} + \varepsilon^{\alpha-\alpha_c}e^{2\sigma C_0 t}.$$

Applying the operators A^ε and B^ε to the equation satisfied by w^ε, we find:

$$\left(i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\partial_x^2 - V(x)\right)A^\varepsilon w^\varepsilon = \sqrt{\varepsilon}\left(V'(x) - V'(x(t))\right)w^\varepsilon - A^\varepsilon L^\varepsilon + A^\varepsilon N^\varepsilon,$$
$$\left(i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\partial_x^2 - V(x)\right)B^\varepsilon w^\varepsilon = \varepsilon A^\varepsilon w^\varepsilon - B^\varepsilon L^\varepsilon + B^\varepsilon N^\varepsilon.$$

We observe that in view of (2.2), A^ε acts on gauge invariant non linearities like a derivative. Therefore, so long as (2.3) holds,

$$\begin{aligned}\|A^\varepsilon(t)N^\varepsilon(t)\|_{L^2(\mathbf{R})} &\lesssim \varepsilon^\alpha\left(\|\varphi^\varepsilon_{\rm lin}(t)\|^{2\sigma}_{L^\infty(\mathbf{R})} + \|w^\varepsilon(t)\|^{2\sigma}_{L^\infty(\mathbf{R})}\right)\|A^\varepsilon(t)\varphi^\varepsilon_{\rm lin}(t)\|_{L^2(\mathbf{R})}\\ &\quad + \varepsilon^\alpha\left(\|\varphi^\varepsilon_{\rm lin}(t)\|^{2\sigma}_{L^\infty(\mathbf{R})} + \|w^\varepsilon(t)\|^{2\sigma}_{L^\infty(\mathbf{R})}\right)\|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2(\mathbf{R})}\\ &\lesssim \varepsilon^{\alpha-\sigma/2}e^{2\sigma C_0 t}\left(e^{Ct} + \|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2(\mathbf{R})}\right).\end{aligned}$$

Similarly, we obtain

$$\|B^\varepsilon(t)N^\varepsilon(t)\|_{L^2(\mathbf{R})} \lesssim \varepsilon^{\alpha-\sigma/2}e^{2\sigma C_0 t}\left(e^{Ct} + \|B^\varepsilon(t)w^\varepsilon(t)\|_{L^2(\mathbf{R})}\right).$$

We infer, thanks to the linear estimates,

$$\begin{aligned}
\|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2(\mathbf{R})} &\lesssim \|B^\varepsilon w^\varepsilon\|_{L^1([0,t];L^2(\mathbf{R}))} + \sqrt{\varepsilon}e^{Ct} \\
&\quad + \varepsilon^{\alpha-\alpha_c}\int_0^t e^{2\sigma C_0 s}\left(e^{Cs} + \|A^\varepsilon(s)w^\varepsilon(s)\|_{L^2(\mathbf{R})}\right)ds, \\
\|B^\varepsilon(t)w^\varepsilon(t)\|_{L^2(\mathbf{R})} &\lesssim \|A^\varepsilon w^\varepsilon\|_{L^1([0,t];L^2(\mathbf{R}))} + \sqrt{\varepsilon}e^{Ct} \\
&\quad + \varepsilon^{\alpha-\alpha_c}\int_0^t e^{2\sigma C_0 s}\left(e^{Cs} + \|B^\varepsilon(s)w^\varepsilon(s)\|_{L^2(\mathbf{R})}\right)ds.
\end{aligned}$$

Gronwall Lemma yields, so long as (2.3) holds:

$$\begin{aligned}
\|w^\varepsilon(t)\|_{\mathcal{H}} &\lesssim \int_0^t \varepsilon^\gamma e^{Cs}\exp\left(C\varepsilon^{\alpha-\alpha_c}\int_s^t e^{2\sigma C_0 s'}ds'\right)ds \\
&\lesssim \exp\left(C\varepsilon^{\alpha-\alpha_c}e^{2\sigma C_0 t}\right)\int_0^t \varepsilon^\gamma e^{Cs}ds \lesssim \exp\left(C\varepsilon^{\alpha-\alpha_c}e^{2\sigma C_0 t}\right)\varepsilon^\gamma e^{Ct},
\end{aligned}$$

where $\gamma = \min(1/2, \alpha-\alpha_c)$. First, we notice that

$$\varepsilon^{\alpha-\alpha_c}e^{2\sigma C_0 t} \leqslant 1 \quad \text{for } 0 \leqslant t \leqslant \frac{\alpha-\alpha_c}{2\sigma C_0}\log\frac{1}{\varepsilon}.$$

Then, setting $\kappa = \frac{\alpha-\alpha_c}{2\sigma C_0}$, Gagliardo-Nirenberg inequality yields, so long as (2.3) holds, with also $t \leqslant \kappa\log\frac{1}{\varepsilon}$, and thanks to the factorization (2.2),

$$\|w^\varepsilon(t)\|_{L^\infty(\mathbf{R})} \lesssim \frac{1}{\varepsilon^{1/4}}\|w^\varepsilon(t)\|^{1/2}_{L^2(\mathbf{R})}\|A^\varepsilon(t)w^\varepsilon(t)\|^{1/2}_{L^2(\mathbf{R})} \lesssim \varepsilon^{\gamma-1/4}e^{Ct}.$$

This is enough to show that the bootstrap argument (2.3) works provided the time variable is restricted to

$$C\varepsilon^\gamma e^{Ct} \leqslant 1,$$

that is, $0 \leqslant t \leqslant C\log\frac{1}{\varepsilon}$ for some $C>0$ independent of ε. Proposition 1.5 follows in the case $d=1$.

To prove Proposition 1.5 when $d \geqslant 2$, one can use Strichartz estimates. This approach is more technical. Since the case $\alpha > \alpha_c$ does not seem the most interesting one, and since we will use Strichartz estimates in the fully nonlinear case, we choose not to present the proof of Proposition 1.5 when $d \geqslant 2$.

3. Fully nonlinear case: bounded time intervals

In this section, we prove Proposition 1.7. This gives us the opportunity to introduce some technical tools which will be used to study large time asymptotics.

3.1. Strichartz estimates.

Definition 3.1. *A pair (q,r) is* **admissible** *if $2 \leqslant r \leqslant \frac{2d}{d-2}$ (resp. $2 \leqslant r \leqslant \infty$ if $d=1$, $2 \leqslant r < \infty$ if $d=2$) and*

$$\frac{2}{q} = \delta(r) := d\left(\frac{1}{2}-\frac{1}{r}\right).$$

Following [20, 40, 27], Strichartz estimates are available for the Schrödinger equation without external potential. Thanks to the construction of the parametrix performed in [15, 16], similar results are available in the presence of an external

satisfying Assumption 1.1 (V could even depend on time). Denote by $U^\varepsilon(t)$ the semi-group associated to $-\frac{\varepsilon^2}{2}\Delta + V$: $\phi^\varepsilon(t,x) = U^\varepsilon(t)\phi_0(x)$ if

$$i\varepsilon\partial_t\phi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\phi^\varepsilon = V\phi^\varepsilon \quad ; \quad \phi^\varepsilon(0,x) = \phi_0(x).$$

From [15], it satisfies the following properties:

- The map $t \mapsto U^\varepsilon(t)$ is strongly continuous.
- $U^\varepsilon(t)U^\varepsilon(s) = U^\varepsilon(t+s)$.
- $U^\varepsilon(t)^* = U^\varepsilon(t)^{-1} = U^\varepsilon(-t)$.
- $U^\varepsilon(t)$ is unitary on L^2: $\|U^\varepsilon(t)\phi\|_{L^2(\mathbf{R}^d)} = \|\phi\|_{L^2(\mathbf{R}^d)}$.
- Dispersive properties: there exist $\delta, C > 0$ independent of $\varepsilon \in]0,1]$ such that for all $t \in \mathbf{R}$ with $|t| \leqslant \delta$,
$$\|U^\varepsilon(t)\|_{L^1(\mathbf{R}^d)\to L^\infty(\mathbf{R}^d)} \leqslant \frac{C}{(\varepsilon|t|)^{d/2}}.$$

We infer the following result, from [27]:

Lemma 3.2 (Scaled Strichartz inequalities). *Let (q,r), (q_1,r_1) and (q_2,r_2) be admissible pairs. Let I be some finite time interval.*
1. *There exists $C = C(r,|I|)$ independent of ε, such that for all $\phi \in L^2(\mathbf{R}^d)$,*

$$\varepsilon^{1/q}\left\|U^\varepsilon(\cdot)\phi\right\|_{L^q(I;L^r(\mathbf{R}^d))} \leqslant C\|\phi\|_{L^2(\mathbf{R}^d)}. \tag{3.1}$$

2. *If I contains the origin, $0 \in I$, denote*

$$D_I^\varepsilon(F)(t,x) = \int_{I\cap\{s\leqslant t\}} U^\varepsilon(t-s)F(s,x)ds.$$

There exists $C = C(r_1,r_2,|I|)$ independent of ε such that for all $F \in L^{q_2'}(I;L^{r_2'})$,

$$\varepsilon^{1/q_1+1/q_2}\left\|D_I^\varepsilon(F)\right\|_{L^{q_1}(I;L^{r_1}(\mathbf{R}^d))} \leqslant C\left\|F\right\|_{L^{q_2'}\left(I;L^{r_2'}(\mathbf{R}^d)\right)}. \tag{3.2}$$

3.2. **Proof of Proposition 1.7.** Denote the error term by $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon$, where φ^ε is now given by (1.18), and $u \in C(\mathbf{R};\Sigma)$ satisfies (1.17). This remainder solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = Vw^\varepsilon - \mathcal{L}^\varepsilon + \lambda\varepsilon^{\alpha_c}\left(|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right) \quad ; \quad w^\varepsilon_{|t=0} = 0, \tag{3.3}$$

where

$$\mathcal{L}^\varepsilon(t,x) = \left(V(x) - T_2\left(x,x(t)\right)\right)\varphi^\varepsilon(t,x) \tag{3.4}$$

is the nonlinear analogue of L^ε given by (2.1). Duhamel's formula for w^ε reads

$$\begin{aligned} w^\varepsilon(t+\tau) = U^\varepsilon(\tau)w^\varepsilon(t) &+ i\varepsilon^{-1}\int_t^{t+\tau} U^\varepsilon(t+\tau-s)\mathcal{L}^\varepsilon(s)ds \\ &- i\lambda\varepsilon^{\alpha_c-1}\int_t^{t+\tau} U^\varepsilon(t+\tau-s)\left(|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right)(s)ds. \end{aligned}$$

Introduce the following Lebesgue exponents:

$$\theta = \frac{2\sigma(2\sigma+2)}{2-(d-2)\sigma} \quad ; \quad q = \frac{4\sigma+4}{d\sigma} \quad ; \quad r = 2\sigma+2.$$

Then (q,r) is admissible, and

$$\frac{1}{q'} = \frac{2\sigma}{\theta} + \frac{1}{q} \quad ; \quad \frac{1}{r'} = \frac{2\sigma}{r} + \frac{1}{r}.$$

Let $t \geqslant 0$, $\tau > 0$ and $I = [t, t+\tau]$. Lemma 3.2 yields

$$\|w^\varepsilon\|_{L^q(I;L^r)} \lesssim \varepsilon^{-1/q}\|w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1-1/q}\|\mathcal{L}^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{\alpha_c-1-2/q}\left\||\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right\|_{L^{q'}(I;L^{r'})}.$$

In view of the pointwise estimate

$$\left||\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right| \lesssim \left(|w^\varepsilon|^{2\sigma} + |\varphi^\varepsilon|^{2\sigma}\right)|w^\varepsilon|,$$

we infer

$$\|w^\varepsilon\|_{L^q(I;L^r)} \lesssim \varepsilon^{-1/q}\|w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1-1/q}\|\mathcal{L}^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{\alpha_c-1-2/q}\left(\|w^\varepsilon\|^{2\sigma}_{L^\theta(I;L^r)} + \|\varphi^\varepsilon\|^{2\sigma}_{L^\theta(I;L^r)}\right)\|w^\varepsilon\|_{L^q(I;L^r)}. \tag{3.5}$$

Thanks to [7], we know that the rescaled functions for ψ^ε and φ^ε, are such that $u^\varepsilon, u \in C(\mathbf{R};\Sigma)$, with estimates which are uniform in $\varepsilon \in]0,1]$. Typically, for all $T > 0$, there exists $C(T)$ independent of ε such that

$$\|Pu^\varepsilon\|_{L^\infty([0,T];L^2)} + \|Pu\|_{L^\infty([0,T];L^2)} \leqslant C(T), \quad P \in \{\mathrm{Id}, \nabla, x\}.$$

In terms of ψ^ε and φ^ε, this yields

$$\|\mathcal{P}^\varepsilon\psi^\varepsilon\|_{L^\infty([0,T];L^2)} + \|\mathcal{P}^\varepsilon\varphi^\varepsilon\|_{L^\infty([0,T];L^2)} \leqslant C(T), \quad \mathcal{P}^\varepsilon \in \{\mathrm{Id}, A^\varepsilon, B^\varepsilon\}. \tag{3.6}$$

The formula (2.2) and Gagliardo–Nirenberg inequality yield, if $0 \leqslant \delta(p) < 1$,

$$\|f\|_{L^p(\mathbf{R}^d)} \leqslant \frac{C(p)}{\varepsilon^{\delta(p)/2}}\|f\|^{1-\delta(p)}_{L^2(\mathbf{R}^d)}\|A^\varepsilon(t)f\|^{\delta(p)}_{L^2(\mathbf{R}^d)}, \quad \forall f \in H^1(\mathbf{R}^d), \quad \forall t \in \mathbf{R}. \tag{3.7}$$

We infer that there exists $C(T)$ independent of ε such that

$$\|\psi^\varepsilon(t)\|_{L^r(\mathbf{R}^d)} + \|\varphi^\varepsilon(t)\|_{L^r(\mathbf{R}^d)} \leqslant C(T)\varepsilon^{-\delta(r)/2}, \quad \forall t \in [0,T]. \tag{3.8}$$

Recalling that $I = [t, t+\tau]$, we deduce from (3.5):

$$\|w^\varepsilon\|_{L^q(I;L^r)} \lesssim \varepsilon^{-1/q}\|w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1-1/q}\|\mathcal{L}^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{\alpha_c-1-2/q}\tau^{2\sigma/\theta}\varepsilon^{-\sigma\delta(r)}\|w^\varepsilon\|_{L^q(I;L^r)}.$$

Since (q,r) is admissible, we compute

$$\alpha_c - 1 - \frac{2}{q} - \sigma\delta(r) = \frac{d\sigma}{2} - \frac{2\sigma+2}{q} = 0, \tag{3.9}$$

hence

$$\|w^\varepsilon\|_{L^q(I;L^r)} \lesssim \varepsilon^{-1/q}\|w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1-1/q}\|\mathcal{L}^\varepsilon\varphi^\varepsilon\|_{L^1(I;L^2)} + \tau^{2\sigma/\theta}\|w^\varepsilon\|_{L^q(I;L^r)}. \tag{3.10}$$

Choosing τ sufficiently small, and repeating this manipulation a finite number of times to cover the time interval $[0,T]$, we obtain

$$\|w^\varepsilon\|_{L^q([0,T];L^r)} \lesssim \varepsilon^{-1/q}\|w^\varepsilon\|_{L^1([0,T];L^2)} + \varepsilon^{-1-1/q}\|\mathcal{L}^\varepsilon\|_{L^1([0,T];L^2)}. \tag{3.11}$$

Using Strichartz estimates again and (3.8), we have, with $J = [0,t]$ and $0 \leqslant t \leqslant T$,

$$\begin{aligned}\|w^\varepsilon\|_{L^\infty(J;L^2)} &\lesssim \varepsilon^{-1}\|\mathcal{L}^\varepsilon\|_{L^1(J;L^2)} + \varepsilon^{\alpha_c-1-1/q}\left\||\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right\|_{L^{q'}(J;L^{r'})}\\ &\lesssim \|w^\varepsilon\|_{L^1(J;L^2)} + \varepsilon^{-1}\|\mathcal{L}^\varepsilon\|_{L^1(J;L^2)}\\ &\quad + \varepsilon^{\alpha_c-1-1/q}\varepsilon^{-1-1/q-\sigma\delta(r)}\|\mathcal{L}^\varepsilon\|_{L^1(J;L^2)}\\ &\lesssim \|w^\varepsilon\|_{L^1(J;L^2)} + \varepsilon^{-1}\|\mathcal{L}^\varepsilon\|_{L^1(J;L^2)},\end{aligned}$$

where the last estimate stems from (3.9). We have the pointwise control

$$|\mathcal{L}^\varepsilon| \lesssim |x-x(t)|^3|\varphi^\varepsilon(t,x)| = \varepsilon^{3/2}\varepsilon^{-d/4}\left(|y|^3|u(t,y)|\right)\Big|_{y=\frac{x-x(t)}{\sqrt{\varepsilon}}}.$$

We infer

$$\varepsilon^{-1}\|\mathcal{L}^\varepsilon\|_{L^1([0,T];L^2(\mathbf{R}^d))} \lesssim \sqrt{\varepsilon}\||y|^3u(t,y)\|_{L^1([0,T];L^2(\mathbf{R}^d))},$$

and the first part of Proposition 1.7 follows from Gronwall lemma.

To establish a control of the $\mathcal{H}$-norm, we notice that in view of (2.2), we have, for $\mathcal{P}^\varepsilon \in \{A^\varepsilon, B^\varepsilon\}$,

$$\mathcal{P}^\varepsilon\left(|\phi^\varepsilon|^{2\sigma}\phi^\varepsilon\right) \approx |\phi^\varepsilon|^{2\sigma}\mathcal{P}^\varepsilon\phi^\varepsilon,$$

where the symbol "$\approx$" is here to recall the abuse of notation when $\mathcal{P}^\varepsilon = A^\varepsilon$ (there should be two terms on the right hand side, with coefficients). Lemma 2.2 shows that we have

$$\begin{aligned}\left(i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta - V\right)A^\varepsilon w^\varepsilon &= \sqrt{\varepsilon}\left(\nabla V(x) - \nabla V\left(x(t)\right)\right)w^\varepsilon - A^\varepsilon\mathcal{L}^\varepsilon\\ &\quad + \lambda\varepsilon^{\alpha_c}A^\varepsilon\left(|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right).\end{aligned}$$

The first term of the right hand side is controlled pointwise by $C\varepsilon|B^\varepsilon w^\varepsilon|$. The L^2-norm of the second term is estimated by

$$\|A^\varepsilon(t)\mathcal{L}^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} \lesssim \varepsilon^{3/2}\left(\||y|^2v(t,y)\|_{L^2(\mathbf{R}^d)} + \||y|^3\nabla v(t,y)\|_{L^2(\mathbf{R}^d)}\right).$$

Finally, we have

$$\begin{aligned}A^\varepsilon\left(|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}\varphi^\varepsilon\right) &\approx |\psi^\varepsilon|^{2\sigma}A^\varepsilon\psi^\varepsilon - |\varphi^\varepsilon|^{2\sigma}A^\varepsilon\varphi^\varepsilon\\ &\approx |w^\varepsilon+\varphi^\varepsilon|^{2\sigma}(A^\varepsilon w^\varepsilon + A^\varepsilon\varphi^\varepsilon) - |\varphi^\varepsilon|^{2\sigma}A^\varepsilon\varphi^\varepsilon\\ &\approx |w^\varepsilon+\varphi^\varepsilon|^{2\sigma}A^\varepsilon w^\varepsilon + \left(|w^\varepsilon+\varphi^\varepsilon|^{2\sigma} - |\varphi^\varepsilon|^{2\sigma}\right)A^\varepsilon\varphi^\varepsilon. \end{aligned}\tag{3.12}$$

The first term of (3.12) is handled like in the first step. For the second term, we have, since $\sigma > 1/2$,

$$\left||w^\varepsilon+\varphi^\varepsilon|^{2\sigma} - |\varphi^\varepsilon|^{2\sigma}\right| \lesssim \left(|w^\varepsilon|^{2\sigma-1} + |\varphi^\varepsilon|^{2\sigma-1}\right)|w^\varepsilon|.$$

Following the same lines as for the L^2 estimate, we find

$$\begin{aligned}\|A^\varepsilon w^\varepsilon\|_{L^q(I;L^r)} &\lesssim \varepsilon^{-1/q}\|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1/q}\|B^\varepsilon w^\varepsilon\|_{L^1(I;L^2)}\\ &\quad+\varepsilon^{-1-1/q}\|A^\varepsilon\mathcal{L}^\varepsilon\|_{L^1(I;L^2)} + \tau^{2\sigma/\theta}\left\|A^\varepsilon w^\varepsilon\right\|_{L^q(I;L^r)} + \tau^{2\sigma/\theta}\left\|w^\varepsilon\right\|_{L^q(I;L^r)},\end{aligned}$$

by using the estimate

$$\|A^\varepsilon(t)\varphi^\varepsilon(t)\|_{L^r(\mathbf{R}^d)} \lesssim \varepsilon^{-\delta(r)/2}\|A^\varepsilon(t)\varphi^\varepsilon(t)\|^{1-\delta(r)}_{L^2(\mathbf{R}^d)}\|A^\varepsilon(t)^2\varphi^\varepsilon(t)\|^{\delta(r)}_{L^2(\mathbf{R}^d)},$$

and the remark

$$\|A^\varepsilon(t)^2\varphi^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} = \|\nabla^2u(t)\|_{L^2(\mathbf{R}^d)}.$$

Since $\sigma > 1/2$, the nonlinearity $z \mapsto |z|^{2\sigma}z$ is twice differentiable, and one can prove $u \in C(\mathbf{R}; H^2(\mathbf{R}^d))$ ([7]). Using (3.11) and the same argument as in the first step, we infer

$$\begin{aligned}\|A^\varepsilon w^\varepsilon\|_{L^q(I;L^r)} &\lesssim \varepsilon^{-1/q}\|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1/q}\|B^\varepsilon w^\varepsilon\|_{L^1(I;L^2)}\\ &\quad+ \varepsilon^{-1-1/q}\|A^\varepsilon\mathcal{L}^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{-1-1/q}\left\|\mathcal{L}^\varepsilon\right\|_{L^1(I;L^2)},\end{aligned}$$

hence, using Strichartz estimates again,

$$\begin{aligned}\|A^\varepsilon w^\varepsilon\|_{L^\infty(I;L^2)} &\lesssim \|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2} + \|B^\varepsilon w^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{-1}\|A^\varepsilon \mathcal{L}^\varepsilon\|_{L^1(I;L^2)} \\ &\quad + \varepsilon^{1/q}\|A^\varepsilon w^\varepsilon\|_{L^q(I;L^r)} + \varepsilon^{1/q}\|w^\varepsilon\|_{L^q(I;L^r)} \\ &\lesssim \|A^\varepsilon(t)w^\varepsilon(t)\|_{L^2} + \|B^\varepsilon w^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{-1}\|A^\varepsilon \mathcal{L}^\varepsilon\|_{L^1(I;L^2)} \\ &\quad + \varepsilon^{-1}\|\mathcal{L}^\varepsilon\|_{L^1(I;L^2)} + \varepsilon^{1/q}\|w^\varepsilon\|_{L^q(I;L^r)}.\end{aligned}$$

Since we have similar estimates for $B^\varepsilon w^\varepsilon$, we end up with

$$\begin{aligned}\|A^\varepsilon w^\varepsilon\|_{L^\infty(J;L^2)} + \|B^\varepsilon w^\varepsilon\|_{L^\infty(J;L^2)} &\lesssim \|A^\varepsilon w^\varepsilon\|_{L^1(J;L^2)} + \|B^\varepsilon w^\varepsilon\|_{L^1(J;L^2)} \\ &\quad + \varepsilon^{-1}\sum_{\mathcal{P}^\varepsilon\in\{\mathrm{Id},A^\varepsilon,B^\varepsilon\}} \|\mathcal{P}^\varepsilon\mathcal{L}^\varepsilon\|_{L^1(J;L^2)} \\ &\lesssim \|A^\varepsilon w^\varepsilon\|_{L^1(J;L^2)} + \|B^\varepsilon w^\varepsilon\|_{L^1(J;L^2)} + \sqrt{\varepsilon}.\end{aligned}$$

Proposition 1.7 then follows from Gronwall lemma.

4. Fully nonlinear case: proof of Theorem 1.12

To prove Theorem 1.12, the strategy consists in examining more carefully the dependence of the $L^\theta L^r$-norms with respect to time in the previous proof. Also, since $(Exp)_4$ concerns only u, not u^ε, we need a bootstrap argument in order to use the same control for the error term w^ε as for the approximate solution φ^ε. This control carries on the $L^r(\mathbf{R}^d)$-norms, for fixed t. By $(Exp)_1$, the relation

$$\|A^\varepsilon(t)\varphi^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} = \|\nabla u(t)\|_{L^2(\mathbf{R}^d)},$$

and the modified Gagliardo–Nirenberg inequality (3.7), we have the following estimate, for all time:

$$\|\varphi^\varepsilon(t)\|_{L^r(\mathbf{R}^d)} \lesssim \varepsilon^{-\delta(r)/2}e^{\kappa t}. \tag{4.1}$$

We will use the following bootstrap argument:

$$\|w^\varepsilon(t)\|_{L^r(\mathbf{R}^d)} \leqslant \varepsilon^{-\delta(r)/2}e^{\kappa t}, \quad t\in[0,T], \tag{4.2}$$

with the same constant κ as in (4.1) to fix the ideas. By Proposition 1.7, for any $T>0$ independent of ε, (4.2) is satisfied provided $0<\varepsilon\leqslant\varepsilon(T)$. By this argument only, it may very well happen that $\varepsilon(T)\to 0$ as $T\to+\infty$. The goal of the bootstrap argument is to show that we can take $T^\varepsilon = C\log\log\frac{1}{\varepsilon}$ for some $C>0$ independent of ε, provided that ε is sufficiently small.

The key step to analyze is the absorption argument, which made it possible to infer (3.11) from (3.10). We resume the computations of §3.2 from the estimate (3.5). Rewrite this estimate with $I=[t,t+\tau]$, $t,\tau\geqslant 0$:

$$\begin{aligned}\|w^\varepsilon\|_{L^q(I;L^r)} &\lesssim \varepsilon^{-1/q}\|w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1-1/q}\|\mathcal{L}_\varepsilon\|_{L^1(I;L^2)} \\ &\quad + \varepsilon^{\alpha_c-1-2/q}\left(\|w^\varepsilon\|^{2\sigma}_{L^\theta(I;L^r)} + \|\varphi^\varepsilon\|^{2\sigma}_{L^\theta(I;L^r)}\right)\|w^\varepsilon\|_{L^q(I;L^r)}.\end{aligned}$$

For simplicity, assume $\tau\leqslant 1$: (4.1) and (4.2) yield, in view of (3.9),

$$\begin{aligned}\|w^\varepsilon\|_{L^q(I;L^r)} \leqslant M\Big(&\varepsilon^{-1/q}\|w^\varepsilon(t)\|_{L^2} + \varepsilon^{-1-1/q}\|\mathcal{L}_\varepsilon\|_{L^1(I;L^2)} \\ &+ \tau^{1/\theta}e^{2\sigma\kappa t}\|w^\varepsilon\|_{L^q(I;L^r)}\Big),\end{aligned}$$

for some constant M independent of ε, $t \geqslant 0$ and $0 \leqslant \tau \leqslant 1$. In order for the last term to be absorbed by the left hand side, we have to assume

$$M\tau^{1/\theta}e^{2\sigma\kappa t} \leqslant \frac{1}{2}, \quad \text{that is,} \quad \tau \leqslant Ce^{-Ct}$$

for some C independent of ε, $t \geqslant 0$ and $0 \leqslant \tau \leqslant 1$. Proceeding with the same argument as in §3.2, we come up with:

$$\begin{aligned}\|w^\varepsilon\|_{L^\infty([0,t];L^2)} &\lesssim \int_0^t e^{Cs}\|w^\varepsilon\|_{L^\infty([0,s];L^2)}ds + \varepsilon^{-1}\int_0^t e^{Cs}\|\mathcal{L}_\varepsilon(s)\|_{L^2}ds \\ &\lesssim \int_0^t e^{Cs}\|w^\varepsilon\|_{L^\infty([0,s];L^2)}ds + \sqrt{\varepsilon}e^{Ct},\end{aligned}$$

where we have used $(Exp)_3$. Gronwall lemma yields:

$$\|w^\varepsilon\|_{L^\infty([0,t];L^2)} \lesssim \sqrt{\varepsilon}\exp\left(C\exp(Ct)\right) \lesssim \sqrt{\varepsilon}\exp\left(\exp(2Ct)\right).$$

Mimicking the computations of §3.2, we have, thanks to $(Exp)_4$ and so long as (4.2) holds,

$$\|A^\varepsilon w^\varepsilon\|_{L^\infty([0,t];L^2)} + \|B^\varepsilon w^\varepsilon\|_{L^\infty([0,t];L^2)} \lesssim \sqrt{\varepsilon}\exp\left(\exp(Ct)\right).$$

To conclude, we check that (4.2) holds for $t \leqslant c\log\log\frac{1}{\varepsilon}$, provided c is sufficiently small. Gagliardo–Nirenberg inequality (3.7) yields

$$\begin{aligned}\|w^\varepsilon(t)\|_{L^r(\mathbf{R}^d)} &\lesssim \varepsilon^{-\delta(r)/2}\|w^\varepsilon\|^{1-\delta(r)}_{L^\infty([0,t];L^2)}\|A^\varepsilon w^\varepsilon\|^{\delta(r)}_{L^\infty([0,t];L^2)} \\ &\leqslant \mathcal{M}\varepsilon^{-\delta(r)/2}\sqrt{\varepsilon}\exp\left(\exp(Ct)\right).\end{aligned}$$

Therefore, taking ε sufficiently small, (4.2) holds as long as

$$\mathcal{M}\sqrt{\varepsilon}\exp\left(\exp(Ct)\right) \leqslant e^{\kappa t}.$$

We check that for large t and sufficiently small ε, this remains true for $t \leqslant c\log\log\frac{1}{\varepsilon}$, with c possibly small, but independent of $\varepsilon \in]0,\varepsilon_0]$. This completes the proof of Theorem 1.12.

5. Ehrenfest time in the one-dimensional cubic case

As pointed out in the introduction, since we consider nonlinearities of the form $z \mapsto |z|^{2\sigma}z$ with $\sigma \in \mathbf{N}$, the one-dimensional cubic case is special. Not because it is integrable (see Remark 1.6: (1.17) is not completely integrable, unless no approximation is needed to describe the wave packets, $\psi^\varepsilon \equiv \varphi^\varepsilon$), but because it is the only case where the nonlinearity is L^2-subcritical, $\sigma < 2/d$. This case is in contrast with the general case of energy-subcritical nonlinearities: without any other assumption on $Q(t)$ than $Q \in C^\infty(\mathbf{R};\mathbf{R}) \cap L^\infty(\mathbf{R})$, it seems that the only *a priori* control that we have for u, solution to (1.17), is

$$\|u(t)\|_{L^2(\mathbf{R}^d)} = \|a\|_{L^2(\mathbf{R}^d)}, \quad \forall t \in \mathbf{R}. \tag{5.1}$$

A remarkable case where other *a priori* estimates are available is when Q is constant, but in this case, $\psi^\varepsilon \equiv \varphi^\varepsilon$. Otherwise, the most general reasonable assumption seems to be $(Exp)_k$, which has been considered in the previous section. Note also that if $d = 1$, the notations of §3.2 become:

$$\theta = \frac{8}{3} \quad ; \quad q = 8 \quad ; \quad r = 4.$$

So to improve the result of Theorem 1.12, we assume $\sigma = d = 1$ and start with the crucial remark:

Lemma 5.1. *Suppose $\sigma = d = 1$, and for $a \in L^2(\mathbf{R})$, consider $u \in C(\mathbf{R}; L^2(\mathbf{R}))$ the solution to* (1.17). *Then there exists C such that*

$$\|u\|_{L^8([t,t+1];L^4(\mathbf{R}))} \leqslant C\|a\|_{L^2(\mathbf{R})}, \quad \forall t \in \mathbf{R}.$$

Proof. First, recall that since $\sigma = d = 1$ and $a \in L^2(\mathbf{R})$, (1.17) has a unique solution

$$u \in C(\mathbf{R}; L^2(\mathbf{R})) \cap L^8_{\rm loc}\left(\mathbf{R}; L^4(\mathbf{R})\right).$$

In addition, (5.1) holds. Denoting

$$W(t,x) = \frac{1}{2}V''\left(x(t)\right)x^2,$$

it has been established in [7] that since $V'' \in L^\infty(\mathbf{R};\mathbf{R})$, uniform local Strichartz estimates are available for the linear propagator. Following [15, 16], let $U(t,s)$ be such that as $u(t,x) = U(t,s)u_0(x)$ is the solution to

$$i\partial_t u + \frac{1}{2}\Delta u = W(t,x)u \quad ; \quad u(s,x) = u_0(x).$$

Then Lemma 3.2 remains true (with $\varepsilon = 1$) when $U^\varepsilon(t-s)$ is replaced with $U(t,s)$, $t,s \in \mathbf{R}$.

Let $t,\tau \geqslant 0$, with $\tau \leqslant 1$, and denote $I = [t, t+\tau]$. Strichartz inequalities yield:

$$\|u\|_{L^8(I;L^4)} \leqslant C(\tau)\|u(t)\|_{L^2} + C(\tau)\left\||u|^2u\right\|_{L^{8/7}(I;L^{4/3})}.$$

In view of (5.1), and using Hölder inequality after the decomposition

$$\frac{3}{4} = \frac{3}{4} + \frac{1}{\infty} \quad ; \quad \frac{7}{8} = \frac{3}{8} + \frac{1}{2},$$

we infer

$$\|u\|_{L^8(I;L^4)} \leqslant C(\tau)\|a\|_{L^2} + C(\tau)\sqrt{\tau}\|u\|^3_{L^8(I;L^4)}.$$

Since $\tau \leqslant 1$, we may assume that $C(\tau)$ does not depend on τ:

$$\|u\|_{L^8(I;L^4)} \leqslant C\|a\|_{L^2} + C\sqrt{\tau}\|u\|^3_{L^8(I;L^4)}.$$

We use the following standard bootstrap argument, borrowed from [2]:

Lemma 5.2 (Bootstrap argument)**.** *Let $f = f(t)$ be a nonnegative continuous function on $[0,T]$ such that, for every $t \in [0,T]$,*

$$f(t) \leqslant M + \delta f(t)^\theta,$$

where $M, \delta > 0$ and $\theta > 1$ are constants such that

$$M < \left(1 - \frac{1}{\theta}\right)\frac{1}{(\theta\delta)^{1/(\theta-1)}} \quad ; \quad f(0) \leqslant \frac{1}{(\theta\delta)^{1/(\theta-1)}}.$$

Then, for every $t \in [0,T]$, we have

$$f(t) \leqslant \frac{\theta}{\theta - 1}M.$$

Lemma 5.1 follows with $[t,t+1]$ replaced with $[t,t+\tau]$ for $0 < \tau \leqslant \tau_0 \ll 1$. We then cover any interval of the form $[t,t+1]$ by a finite number of intervals of length at most τ_0, and Lemma 5.1 is proved. □

Proof of Theorem 1.13. Like in the previous section, we resume the proof of Proposition 1.7, and pay a more precise attention to the dependence of various constants upon time. We modify the bootstrap argument of §4: in view of Lemma 5.1, (4.2) is replaced by

$$\|w^\varepsilon\|_{L^8([t,t+1];L^4(\mathbf{R}))} \leqslant \varepsilon^{-1/8}\|a\|_{L^2(\mathbf{R})}, \quad \forall t\in[0,T]. \tag{5.2}$$

By Proposition 1.7, for any $T>0$ independent of ε, (5.2) remains true provided $0<\varepsilon\leqslant\varepsilon(T)$. Keeping the notations of §3.2, we have:

$$\theta=\frac{8}{3}\quad ;\quad q=8\quad ;\quad r=4.$$

With $t\geqslant 0$, $\tau\in]0,1]$ and $I=[t,t+\tau]$, (3.5) becomes

$$\begin{aligned}\|w^\varepsilon\|_{L^8(I;L^4)} &\lesssim \varepsilon^{-1/8}\|w^\varepsilon(t)\|_{L^2}+\varepsilon^{-1-1/8}\|\mathcal{L}_\varepsilon\|_{L^1(I;L^2)}\\ &\quad+\varepsilon^{1/4}\left(\|w^\varepsilon\|^2_{L^{8/3}(I;L^4)}+\|\varphi^\varepsilon\|^2_{L^{8/3}(I;L^4)}\right)\|w^\varepsilon\|_{L^8(I;L^4)}\\ &\lesssim \varepsilon^{-1/8}\|w^\varepsilon(t)\|_{L^2}+\varepsilon^{-1-1/8}\|\mathcal{L}_\varepsilon\|_{L^1(I;L^2)}\\ &\quad+\varepsilon^{1/4}\tau^{1/4}\left(\|w^\varepsilon\|^2_{L^{8}(I;L^4)}+\|\varphi^\varepsilon\|^2_{L^{8}(I;L^4)}\right)\|w^\varepsilon\|_{L^8(I;L^4)}\\ &\lesssim \varepsilon^{-1/8}\|w^\varepsilon(t)\|_{L^2}+\varepsilon^{-1-1/8}\|\mathcal{L}_\varepsilon\|_{L^1(I;L^2)}+\tau^{1/4}\|w^\varepsilon\|_{L^8(I;L^4)},\end{aligned}\tag{5.3}$$

where we have used Lemma 5.1 and (5.2). Choosing τ sufficiently small and independent of t, we come up with

$$\begin{aligned}\|w^\varepsilon\|_{L^\infty([0,t];L^2)} &\lesssim \|w^\varepsilon\|_{L^1([0,t];L^2)}+\varepsilon^{-1}\|\mathcal{L}_\varepsilon\|_{L^1([0,t];L^2)}\\ &\lesssim \|w^\varepsilon\|_{L^1([0,t];L^2)}+\sqrt{\varepsilon}\int_0^t e^{Cs}ds,\end{aligned}$$

by $(Exp)_3$. Gronwall lemma yields

$$\|w^\varepsilon\|_{L^\infty([0,t];L^2)}\lesssim\sqrt{\varepsilon}e^{Ct}.$$

Back to (5.3), we infer, with $\tau\ll 1$,

$$\|w^\varepsilon\|_{L^8(I;L^4)}\lesssim\varepsilon^{1/4}e^{Ct}.$$

Therefore, there exists $c>0$ such that (5.2) holds for $T=c\log\frac{1}{\varepsilon}$ provided ε is sufficiently small, hence the first part of Theorem 1.13.

It is then quite straightforward to infer the estimates in $\mathcal{H}$, by rewriting the end of the proof of Proposition 1.7, with (5.2) in mind. □

6. Nonlinear superposition

6.1. **General considerations.** The proof of Proposition 1.14 and Theorem 1.15 follows the same lines as the proof of Proposition 1.7 and Theorem 1.13. The main difference comes from the way one deals with the nonlinearity, since new terms appear. These terms come from the nonlinear interaction between the two profiles φ_1^ε and φ_2^ε. Denote $w^\varepsilon=\psi^\varepsilon-\varphi_1^\varepsilon-\varphi_2^\varepsilon$. It solves

$$i\varepsilon\partial_t w^\varepsilon+\frac{\varepsilon^2}{2}\Delta w^\varepsilon=Vw^\varepsilon-\mathcal{L}^\varepsilon+\lambda\mathcal{N}^\varepsilon\quad ;\quad w^\varepsilon_{|t=0}=0,$$

where we have now

$$\mathcal{L}^\varepsilon(t,x)=\left(V(x)-T_2\left(x,x(t)\right)\right)\left(\varphi_1^\varepsilon(t,x)+\varphi_2^\varepsilon(t,x)\right),$$

and

$$\mathcal{N}^\varepsilon = \varepsilon^{\alpha_c}\left(|w^\varepsilon+\varphi_1^\varepsilon+\varphi_2^\varepsilon|^{2\sigma}(w^\varepsilon+\varphi_1^\varepsilon+\varphi_2^\varepsilon) - |\varphi_1^\varepsilon|^{2\sigma}\varphi_1^\varepsilon - |\varphi_2^\varepsilon|^{2\sigma}\varphi_2^\varepsilon\right).$$

Decompose $\mathcal{N}^\varepsilon$ as the sum of a semilinear term and an interaction source term: $\mathcal{N}^\varepsilon = \mathcal{N}_S^\varepsilon + \mathcal{N}_I^\varepsilon$, where

$$\mathcal{N}_S^\varepsilon = \varepsilon^{\alpha_c}\left(|w^\varepsilon+\varphi_1^\varepsilon+\varphi_2^\varepsilon|^{2\sigma}(w^\varepsilon+\varphi_1^\varepsilon+\varphi_2^\varepsilon) - |\varphi_1^\varepsilon+\varphi_2^\varepsilon|^{2\sigma}(\varphi_1^\varepsilon+\varphi_2^\varepsilon)\right),$$
$$\mathcal{N}_I^\varepsilon = \varepsilon^{\alpha_c}\left(|\varphi_1^\varepsilon+\varphi_2^\varepsilon|^{2\sigma}(\varphi_1^\varepsilon+\varphi_2^\varepsilon) - |\varphi_1^\varepsilon|^{2\sigma}\varphi_1^\varepsilon - |\varphi_2^\varepsilon|^{2\sigma}\varphi_2^\varepsilon\right).$$

We see that the term $\mathcal{N}_S^\varepsilon$ is the exact analogue of the nonlinear term in (3.3), where we have simply replaced φ^ε with $\varphi_1^\varepsilon+\varphi_2^\varepsilon$. We can thus repeat the proofs of Proposition 1.7 and Theorem 1.13, respectively, up to the control of the new source term $\mathcal{N}_I^\varepsilon$ (the linear source term $\mathcal{L}^\varepsilon$ is treated as before). More precisely, we have to estimate

$$\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon\|_{L^1([0,t];L^2(\mathbf{R}^d))}.$$

The first remark consists in noticing that if σ is an integer, $\mathcal{N}_I^\varepsilon$ can be estimated (pointwise) by a sum of terms of the form

$$\varepsilon^{\alpha_c}|\varphi_1^\varepsilon|^{\ell_1}\times|\varphi_2^\varepsilon|^{\ell_2},\quad \ell_1,\ell_2\geqslant 1,\ \ell_1+\ell_2=2\sigma+1.$$

To be more precise, we have the control, for fixed time,

$$\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} \lesssim \varepsilon^{d\sigma/2}\sum_{\ell_1,\ell_2\geqslant 1,\ \ell_1+\ell_2=2\sigma+1}\left\||\varphi_1^\varepsilon|^{\ell_1}\times|\varphi_2^\varepsilon|^{\ell_2}\right\|_{L^2(\mathbf{R}^d)}.$$

We will see below why the right hand side must be expected to be small, when integrated with respect to time. We need to estimate

$$\varepsilon^{d\sigma/2}\left\|(\varphi_1^\varepsilon)^{\ell_1}(\varphi_2^\varepsilon)^{\ell_2}\right\|_{L^2(\mathbf{R}^d)} = \left\|u_1^{\ell_1}\left(t,x-\frac{x_1(t)-x_2(t)}{\sqrt{\varepsilon}}\right)u_2^{\ell_2}(t,x)\right\|_{L^2(\mathbf{R}^d_x)},$$

with $\ell_1,\ell_2\geqslant 1$, $\ell_1+\ell_2=2\sigma+1$. We have the following lemma:

Lemma 6.1. *Suppose $d\leqslant 3$, and σ is an integer. Let $T\in\mathbf{R}$, $0<\gamma<1/2$, and*

$$I^\varepsilon(T)=\{t\in[0,T],\ \ |x_1(t)-x_2(t)|\leqslant\varepsilon^\gamma\}. \tag{6.1}$$

Then, for all $k>d/2$,

$$\frac{1}{\varepsilon}\int_0^T\|\mathcal{N}_I^\varepsilon(t)\|_{\Sigma_\varepsilon}dt \lesssim (M_{k+2}(T))^{2\sigma+1}\left(T\varepsilon^{k(1/2-\gamma)}+|I^\varepsilon(T)|\right)e^{CT},$$

where $M_k(T)=\sup\left\{\|\langle x\rangle^\alpha\partial_x^\beta u_j\|_{L^\infty([0,T];L^2(\mathbf{R}^d))};\quad j\in\{1,2\},\quad |\alpha|+|\beta|\leqslant k\right\}.$

Proof. We observe that for $\eta\in\mathbf{R}^d$,

$$\sup_{x\in\mathbf{R}^d}\left(\langle x\rangle^{-1}\langle x-\eta\rangle^{-1}\right)\lesssim\frac{1}{|\eta|}.$$

With $\eta^\varepsilon(t)=\frac{x_1(t)-x_2(t)}{\sqrt{\varepsilon}}$, we infer (forgetting the sum over ℓ_1,ℓ_2),

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{[0,T]\setminus I^\varepsilon(T)}\|\mathcal{N}_I^\varepsilon(t)\|_{L^2(\mathbf{R})}dt\lesssim\\
&\lesssim\int_{[0,T]\setminus I^\varepsilon(T)}\left\|\langle x-\eta^\varepsilon(t)\rangle^{-k}\langle x\rangle^{-k}\langle x-\eta^\varepsilon(t)\rangle^{-k}u_1^{\ell_1}\left(t,x-\eta^\varepsilon(t)\right)u_2^{\ell_2}(t,x)\right\|_{L^2}\\
&\lesssim\left\|\langle x\rangle^k u_1^{\ell_1}\right\|_{L^\infty([0,T];L^4)}\left\|\langle x\rangle^k u_2^{\ell_2}\right\|_{L^\infty([0,T];L^4)}\int_{[0,T]\setminus I^\varepsilon(T)}\frac{dt}{|\eta^\varepsilon(t)|^k}.
\end{aligned}$$

We have, for $j \in \{1,2\}$,

$$\begin{aligned}\left\| \langle x\rangle^k u_j^{\ell_j}\right\|_{L^\infty([0,T];L^4)} &\leqslant \left\| \langle x\rangle^k u_j\right\|_{L^\infty([0,T];L^4)} \|u_j\|^{\ell_j-1}_{L^\infty([0,T]\times \mathbf{R}^d)}\\ &\lesssim \left\| \langle x\rangle^k u_j\right\|_{L^\infty([0,T];H^1)} \|u_j\|^{\ell_1-1}_{L^\infty([0,T];H^k)} \lesssim M_{k+1}(T)^{\ell_j},\end{aligned}$$

where we have used $H^1(\mathbf{R}^d)\subset L^4(\mathbf{R}^d)$ since $d\leqslant 3$. On the other hand,

$$\int_{[0,T]\setminus I^\varepsilon(T)} \frac{dt}{|\eta^\varepsilon(t)|^k}\lesssim \int_{[0,T]\setminus I^\varepsilon(T)} \frac{\varepsilon^{k/2}}{|x_1(t)-x_2(t)|^k}dt \lesssim \varepsilon^{k(1/2-\gamma)}T.$$

On $I^\varepsilon(T)$, we simply estimate

$$\begin{aligned}\frac{1}{\varepsilon}\int_{I^\varepsilon(T)} \|\mathcal{N}_I^\varepsilon(t)\|_{L^2(\mathbf{R})}dt &\lesssim \|u_1\|^{\ell_1}_{L^\infty([0,T]\times\mathbf{R})}\|u_2\|^{\ell_2-1}_{L^\infty([0,T]\times\mathbf{R})}\|u_2\|_{L^1(I^\varepsilon(T);L^2(\mathbf{R}))}\\ &\lesssim M_k(T)^{2\sigma}|I^\varepsilon(T)|\|u_2\|_{L^\infty([0,T];L^2(\mathbf{R}))}\\ &\lesssim M_k(T)^{2\sigma+1}|I^\varepsilon(T)|.\end{aligned}$$

The L^2 estimate follows, without exponentially growing factor. This factor appears when dealing with the Σ_ε-norm. Typically,

$$\begin{aligned}\|\varepsilon\nabla\varphi_j^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} &\lesssim \sqrt{\varepsilon}\|\nabla u_j(t)\|_{L^2(\mathbf{R}^d)} + |\xi_j(t)|\|u_j(t)\|_{L^2(\mathbf{R}^d)},\\ \|x\varphi_j^\varepsilon(t)\|_{L^2(\mathbf{R}^d)} &\lesssim \sqrt{\varepsilon}\|xu_j(t)\|_{L^2(\mathbf{R}^d)} + |x_j(t)|\|u_j(t)\|_{L^2(\mathbf{R}^d)}.\end{aligned}$$

The result then follows from the above computations, and Lemma 1.2. □

At this stage, the main difficulty is to estimate the length of $I^\varepsilon(t)$. We do this in two cases: bounded t, and large time when $d=1$.

6.2. **Nonlinear superposition in finite time.** In the proof of Proposition 1.7, we have only used the fact that $u^\varepsilon\in C(\mathbf{R};\Sigma)$, with estimates which are independent of ε. Recall that in the case of a single wave packet, ψ^ε and u^ε are related through (1.8): in the case of two wave packets, there is no such natural rescaling. So in the case of two initial wave packets, we are not able to prove uniform estimates for ψ^ε, like in (3.6). Even to prove Proposition 1.14, which is the analogue of Proposition 1.7, we need to use a bootstrap argument. We know that for $j\in\{1,2\}$,

$$\|\varphi_j^\varepsilon(t)\|_{L^r(\mathbf{R}^d)}\leqslant C(T)\varepsilon^{-\delta(r)/2},\quad \forall t\in[0,T].$$

The bootstrap argument is of the form:

$$\|w^\varepsilon(t)\|_{L^r(\mathbf{R}^d)}\leqslant C(T)\varepsilon^{-\delta(r)/2},\quad \forall t\in[0,T],$$

with the same constant $C(T)$ if we wish. Repeating the computations of §3.2, we first have, for $t\in[0,T]$ and so long as the above condition holds,

$$\|w^\varepsilon\|_{L^\infty([0,t];L^2)}\lesssim \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L^1([0,T];L^2)}+\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon\|_{L^1([0,T];L^2)}.$$

As we have seen in §2.1, $(\varepsilon\nabla w^\varepsilon, xw^\varepsilon)$ solves a system which is formally analogous to the system satisfied by $(A^\varepsilon w^\varepsilon, B^\varepsilon w^\varepsilon)$. Therefore, under the bootstrap condition, we come up with

$$\|w^\varepsilon\|_{L^\infty([0,t];\Sigma_\varepsilon)}\lesssim \frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L^1([0,T];\Sigma_\varepsilon)}+\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon\|_{L^1([0,T];\Sigma_\varepsilon)}.$$

We easily estimate

$$\frac{1}{\varepsilon}\|\mathcal{L}^\varepsilon\|_{L^1([0,T];\Sigma_\varepsilon)}\lesssim\sqrt{\varepsilon},$$

so in view of Lemma 6.1, the point is to estimate the length of $I^\varepsilon(T)$.

Lemma 6.2. *For $T>0$ (independent of ε), we have*

$$|I^\varepsilon(T)| = \mathcal{O}\left(\varepsilon^\gamma\right),$$

where $I^\varepsilon(T)$ is defined by (6.1).

Proof. The key remark is that since $(x_1,\xi_1)\neq(x_2,\xi_2)$, the trajectories $x_1(t)$ and $x_2(t)$ may cross only in isolated points: by uniqueness, if $x_1(t)=x_2(t)$, then $\dot x_1(t)\neq\dot x_2(t)$. Therefore, there is only a finite numbers of such points in the interval $[0,T]$:

$$\left(x_1(\cdot)-x_2(\cdot)\right)^{-1}(0)\cap[0,T]=\{t_j\}_{1\leqslant j\leqslant J},\quad \text{where } J=J(T).$$

If we had $J=\infty$, then by compactness of $[0,T]$, a subsequence of $(t_j)_j$ would converge to some $\tau\in[0,T]$, with $x_1(\tau)=x_2(\tau)$. By uniqueness for the Hamiltonian flow, $\dot x_1(\tau)\neq\dot x_2(\tau)$: τ cannot be the limit of times where $x_1(t_j)=x_2(t_j)$.

By uniqueness for the Hamiltonian flow, continuity and compactness, there exists $\delta>0$ such that

$$\inf\{|\dot x_1(t)-\dot x_2(t)|\ ;\ t\in\mathcal{I}(\delta,T)\}=m>0,\quad \text{where } \mathcal{I}(\delta,T)=\bigcup_{j=1}^J[t_j-\delta,t_j+\delta],$$

and there exists $\varepsilon(\delta,T)>0$ such that for $\varepsilon\in]0,\varepsilon(\delta,T)]$, $I^\varepsilon(T)\subset\mathcal{I}(\delta,T)$.

Let $t\in I^\varepsilon(T)\cap[t_j-\delta,t_j+\delta]$. Taylor's formula yields

$$x_1(t)-x_2(t)=x_1(t_j)-x_2(t_j)+(t-t_j)\left(\dot x_1(\tau)-\dot x_2(\tau)\right),\quad \tau\in[t_j-\delta,t_j+\delta].$$

We infer

$$\varepsilon^\gamma\geqslant|x_1(t)-x_2(t)|\geqslant|t-t_j|m,$$

and Lemma 6.2 follows. □

Back to the bootstrap argument, we infer

$$\|w^\varepsilon\|_{L^\infty([0,t];\Sigma_\varepsilon)}\lesssim\sqrt{\varepsilon}+\varepsilon^{k(1/2-\gamma)}+\varepsilon^\gamma.$$

Fix $\gamma\in]0,1/2[$. By taking k sufficiently large in Lemma 6.1, this yields

$$\|w^\varepsilon\|_{L^\infty([0,t];\Sigma_\varepsilon)}\lesssim\varepsilon^\gamma.$$

Gagliardo–Nirenberg inequality yields

$$\|w^\varepsilon(t)\|_{L^r}\lesssim\varepsilon^{-\delta(r)}\|w^\varepsilon(t)\|_{L^2}^{1-\delta(r)}\|\varepsilon\nabla w^\varepsilon(t)\|_{L^2}^{\delta(r)}\lesssim\varepsilon^{-\delta(r)}\|w^\varepsilon\|_{L^\infty([0,t];\Sigma_\varepsilon)}\lesssim\varepsilon^{\gamma-\delta(r)}.$$

To close the argument, we note

$$\varepsilon^{\gamma-\delta(r)}\ll\varepsilon^{-\delta(r)/2}\text{ provided }\varepsilon\ll1\text{ and }\gamma>\frac{\delta(r)}{2}.$$

The last condition is equivalent to $\gamma>\frac{d\sigma}{4\sigma+4}$, which is compatible with $\gamma<1/2$ since the nonlinearity is energy-subcritical.

6.3. **Nonlinear superposition for large time.** Things become more complicated when T is large. We first need to control M_k: this is achieved assuming $(Exp)_k$, and we have

$$M_k(t) \lesssim e^{Ct}.$$

The main point is to estimate $|I_\varepsilon|$. This is achieved thanks to the following proposition, whose proof relies heavily on the fact that the space variable is one-dimensional.

Proposition 6.3. *Under the assumptions of Theorem 1.15, there exist $C, C_0 > 0$ independent of ε such that*

$$|I_\varepsilon(t)| \lesssim \varepsilon^\gamma e^{C_0 t}|E_1 - E_2|^{-2}, \quad 0 \leqslant t \leqslant C \log \frac{1}{\varepsilon}.$$

Proof of Theorem 1.15. Before proving Proposition 6.3, we show why this is enough to infer Theorem 1.15. By Lemma 6.1, we have, if $(Exp)_k$ is satisfied,

$$\frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon\|_{L^1([0,t];\Sigma_\varepsilon)} \lesssim e^{Ct}\left(t\varepsilon^{(k-2)(1/2-\gamma)} + \varepsilon^\gamma e^{C_0 t}\right) \lesssim \left(\varepsilon^{(k-2)(1/2-\gamma)} + \varepsilon^\gamma\right) e^{Ct}.$$

Optimizing in γ, we require $(k-2)(1/2-\gamma) = \gamma$, that is

$$\gamma = \frac{k-2}{2k-2}.$$

We can thus resume the bootstrap arguments as in §4 and §5, respectively. The key is to notice that this works like in the previous paragraph, since

$$\gamma = \frac{k-2}{2k-2} > \frac{\sigma}{4\sigma+4} \quad (k \geqslant 4).$$

This yields Theorem 1.15 $\square$

Proof of Proposition 6.3. We consider $J^\varepsilon(t)$ an interval of maximal length included in $I^\varepsilon(t)$ and $N^\varepsilon(t)$ the number of such intervals. The result comes from the estimate

$$|I^\varepsilon(t)| \leqslant N^\varepsilon(t) \times \max|J^\varepsilon(t)|,$$

with

$$|J^\varepsilon(t)| \lesssim \varepsilon^\gamma e^{Ct}|E_1 - E_2|^{-1}, \tag{6.2}$$

$$N^\varepsilon(t) \lesssim te^{2Ct}|E_1 - E_2|^{-1} \lesssim e^{3Ct}|E_1 - E_2|^{-1}. \tag{6.3}$$

We first prove prove (6.2). Let $t_1, t_2 \in J^\varepsilon(t)$. There exists $t^* \in [t_1, t_2]$ such that

$$|(x_1(t_1) - x_2(t_1)) - (x_1(t_2) - x_2(t_2))| = |t_2 - t_1|\,|\xi_1(t^*) - \xi_2(t^*)|\,,$$

whence

$$|t_1 - t_2| \leqslant |\xi_1(t^*) - \xi_2(t^*)|^{-1} \times 2\varepsilon^\gamma.$$

On the other hand,

$$|\xi_1(t^*) - \xi_2(t^*)| \geqslant ||\xi_1(t^*)| - |\xi_2(t^*)|| \geqslant \frac{\left||\xi_1(t^*)|^2 - |\xi_2(t^*)|^2\right|}{|\xi_1(t^*)| + |\xi_2(t^*)|}.$$

Using

$$\begin{aligned}
|\xi_1(t^*)| + |\xi_2(t^*)| &\lesssim e^{Ct},\\
|\xi_1(t^*)|^2 - |\xi_2(t^*)|^2 &= 2\left(E_1 - E_2 - V(x_1(t^*)) + V(x_2(t^*))\right),\\
|V(x_1(t^*)) - V(x_2(t^*))| &\leqslant \varepsilon^\gamma \mathrm{e}^{Ct},
\end{aligned}$$

we get

$$||\xi_1(t^*)|^2 - |\xi_2(t^*)|^2| \gtrsim |E_1 - E_2| - \varepsilon^\gamma e^{Ct},$$

whence

$$|t_1 - t_2| \lesssim \varepsilon^\gamma \mathrm{e}^{Ct} |E_1 - E_2|^{-1},$$

provided $\varepsilon^\gamma e^{Ct} \ll 1$.

Let us now prove (6.3). We use that as t is large, $N^\varepsilon(t)$ is comparable to the number of distinct intervals of maximal size where $|x_1(t) - x_2(t)| \geqslant \varepsilon^\gamma$. We consider $J'_\varepsilon = [t'_1, t'_2]$ such an interval . We have

$$|x_1(t'_1) - x_2(t'_1)| = |x_1(t'_2) - x_2(t'_2)| = \varepsilon^\gamma, \text{ and } \forall t \in [t'_1, t'_2], \quad |x_1(t) - x_2(t)| \geqslant \varepsilon^\gamma.$$

Therefore, for $t \in [t'_1, t'_2]$, the quantity $x_1(t) - x_2(t)$ has a constant sign: we suppose that $x_1(t) - x_2(t)$ is positive. We then have

$$\xi_1(t'_1) - \xi_2(t'_1) > 0 \text{ and } \xi_1(t'_2) - \xi_2(t'_2) < 0.$$

Using the exponential control of $V'(x_j(t))$ for $j \in \{1, 2\}$, we obtain

$$(\xi_1(t'_1) - \xi_2(t'_1)) - (\xi_1(t'_2) - \xi_2(t'_2)) \lesssim \mathrm{e}^{Ct} |t'_1 - t'_2|.$$

We write

$$\begin{aligned}
\xi_1(t'_1) - \xi_2(t'_1) = |\xi_1(t'_1) - \xi_2(t'_1)| &\geqslant \frac{\left||\xi_1(t'_1)|^2 - |\xi_2(t'_1)|^2\right|}{|\xi_1(t'_1)| + |\xi_2(t'_1)|} \\
&\gtrsim e^{-Ct} \left||\xi_1(t'_1)|^2 - |\xi_2(t'_1)|^2\right|, \\
-\xi_1(t'_2) + \xi_2(t'_2) = |\xi'_1(t_2) - \xi_2(t'_2)| &\geqslant \frac{\left||\xi_1(t'_2)|^2 - |\xi_2(t'_2)|^2\right|}{|\xi_1(t'_2)| + |\xi_2(t'_2)|} \\
&\gtrsim e^{-Ct} \left||\xi_1(t'_2)|^2 - |\xi_2(t'_2)|^2\right|.
\end{aligned}$$

Besides, in view of

$$\begin{aligned}
\frac{1}{2}\left(|\xi_1(t'_1)|^2 - |\xi_2(t'_2)|^2\right) &= E_1 - E_2 - V(x_1(t'_1)) + V(x_2(t'_1)) \\
&= E_1 - E_2 - V'(x^*) \left[x_1(t'_1) - x_2(t'_1)\right]
\end{aligned}$$

with $x^* \in [x_2(t'_1), x_1(t'_1)]$, we have

$$|V'(x^*) \left[x_1(t'_1) - x_2(t'_1)\right]| \lesssim e^{Ct} \left[x_1(t'_1) - x_2(t'_1)\right] \lesssim \varepsilon^\gamma e^{Ct}.$$

Therefore, if $\varepsilon^\gamma e^{Ct} \ll 1$, we have $E_1 - E_2 > 0$ and

$$\frac{1}{2}\left||\xi_1(t'_1)|^2 - |\xi_2(t'_)|^2\right| \geqslant \frac{1}{2}(E_1 - E_2).$$

The same holds for t'_2, which yields

$$(\xi_1(t'_1) - \xi_2(t'_1)) - (\xi_1(t'_2) - \xi_2(t'_2)) \gtrsim e^{-Ct}(E_1 - E_2),$$

whence the existence of a constant $c > 0$ such that

$$|t'_1 - t'_2| \geqslant ce^{-2CT}(E_1 - E_2) \text{ and } |J'_\varepsilon| \geqslant c\,\mathrm{e}^{-2CT}(E_1 - E_2).$$

The number $\tilde{N}^\varepsilon(t)$ of intervals of the type J'_ε satisfies

$$\tilde{N}^\varepsilon(t) \times ce^{-2Ct}(E_1 - E_2) \leqslant t$$

whence the second point of the claim. $\square$

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